Pseudo-Differential Operators
Theory and Applications
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Geni Gupur

Functional Analysis Methods for Reliability Models

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Functional Analysis Methods for Reliability Models

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Contents

Preface

People are often concerned with the reliability of products they use and of the friends and associates with whom they interact daily in a myriad of ways. Although the term "reliability" may not occur in the description of a given product or interaction, it is often the attribute that the individual means in expression of their concern. Schedule delays, inconvenience, customer dissatisfaction, and loss of prestige and even weakening of national security are common examples of results that are caused by unreliability of systems and individuals. Thus, both the government and private industry have had to recognize the importance of understanding and strengthening the concept of reliability. The growth, recognition and definitization of the reliability function were given much impetus during the 1960s. During those processes, three main technical areas of reliability evolved:

(1) reliability engineering;
(2) operations analysis;
(3) reliability mathematics.

Each of these areas developed its own body of knowledge. This book is focused on the mathematics of reliability.

I have written this book on the basis of my previous work, both alone and in collaboration with colleagues. Our main goal is to introduce readers to functional analysis methods, in particular, time-dependent analysis, for reliability models, By using theory and methods of functional analysis, we discuss a model of a transfer line consisting of a reliable machine, an unreliable machine and a storage buffer with finite, or infinite, capacity. These are described by a group of integro-differential equations with integral boundary conditions, which comprise a special part of the theory of partial differential equations. As we introduce the relevant partial differential equations, we discuss the existence of a unique positive time-dependent solution and asymptotic behavior of that solution as time tends to infinity. Moreover, we discuss asymptotic behavior of some reliability indices.

The book is divided into five chapters.

Chapter 1 deals with the theory of functional analysis with the objective of providing limited, yet adequate, mathematical tools for analysis of most reliability models.

Chapter 2 provides a brief history of reliability theory and formally states the problems that we have chosen to study.

Chapter 3 is devoted to the dynamic analysis of a transfer line consisting of a reliable machine, an unreliable machine, and a storage buffer with finite capacity, which is described by a finite number of partial differential equations with integral boundary conditions.

Chapter 4 introduces a transfer line consisting of a reliable machine, an unreliable machine and a storage buffer with infinite capacity, which is described by an infinite number of partial differential equations with integral boundary conditions.

Chapter 5 explores some further research problems.

The most interesting mathematics in this book is the detailed analysis of spectra of infinite-dimensional tridiagonal matrices, which can be considered as interesting, albeit special, pseudo-differential operators.

I take great pleasure in acknowledging the generous support provided by Professor Ghalip Mohammed and M.W. Wong at York University, Professor Heinrich Begehr at the Free University of Berlin, Professor Rainer Nagel at Tübingen University and professor Guang-tian Zhu and Bao-zhu Guo at the Chinese Academy of Sciences. I am most grateful to the following granting authorities for their support of our research work: The Natural Science Foundation of China (No: 10861011) and DAAD (German Academic Exchange Service).

Finally, I wish to express my enormous and sincere appreciation to my family.

Despite my serious effort to catch them all, errors undoubtedly remain. I take full responsibility for these, and will greatly appreciate receiving information concerning any errors that readers notice.

Geni Gupur

Chapter 1

C_0-Semigroups of Linear Operators and Cauchy Problems

One-parameter semigroups of linear operators have a simple elegant theory together with a wide variety of applications, covering a broad area. It is essentially complete in that it answers the basic questions of existence, uniqueness, computability and asymptotic stability concerning solutions of linear evolution equations.

The aim of this chapter is to introduce the solution and its asymptotic behavior of problems involving an abstract differential equation in a Banach space, which will be used in later chapters.

This chapter consists of three sections: preliminaries, well-posedness of an abstract Cauchy problem and asymptotic behavior of its solution. In the first section, we will introduce the main concepts and results in the functional analysis theory that will be applied in later parts. The second section involves well-posedness of an abstract Cauchy problem, in which we will state how to prove the existence and uniqueness of the time-dependent solution of an abstract Cauchy problem in a Banach space through C_0-semigroups. In the last section, we will discuss how to study asymptotic behavior of the time-dependent solution of an abstract Cauchy problem by using stability analysis of C_0-semigroups.

1.1 Preliminaries

In this part we will introduce the main concepts and results in functional analysis that will be used in later parts. In this part we mainly refer to Yosida [115], Xia et al. [108], Taylor et al. [104], Gupur [57], Arendt et al. [6], Kelley [72] and Belleni-Morente et al. [8].

Definition 1.1. A set X is called a linear space over a field $\mathbb{K}$ if the following conditions are satisfied:

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1. X is an Abelian group (written additively).
2. A scalar multiplication is defined: to every element $x \in X$ and each $\alpha \in \mathbb{K}$ there is associated an element of X, denoted by αx, such that we have:
   (1) $\alpha(x+y) = \alpha x + \alpha y$ for $\alpha \in \mathbb{K},\ x, y \in X$;
   (2) $(\alpha+\beta)x = \alpha x + \beta y$ for $\alpha, \beta \in \mathbb{K},\ x \in X$;
   (3) $(\alpha\beta)x = \alpha(\beta x)$ for $\alpha, \beta \in \mathbb{K},\ x \in X$;
   (4) $1 \cdot x = x$ for the unit element 1 of the field $\mathbb{K}$.

If $\mathbb{K} = \mathbb{R}$ (the real number field), then X is said to be a real linear space. If $\mathbb{K} = \mathbb{C}$ (the complex number field), then X is called a complex linear space.

Definition 1.2. Let X be a linear space over a field $\mathbb{K}$. A subset D of X is called a linear subspace if $\alpha x + \beta y \in D$ for $\forall \alpha, \beta \in \mathbb{K};\ x, y \in X$.

Definition 1.3. Let X and Y be linear spaces over the same field $\mathbb{K}$. A mapping $T : x \to y = T(x) \stackrel{\text{set}}{=\!=} Tx$ defined on a linear subspace D of X and taking values in Y is said to be a linear operator, if

$$T(\alpha x + \beta y) = \alpha Tx + \beta y,\ \forall \alpha, \beta \in \mathbb{K},\ x, y \in X.$$

$D(T) \stackrel{\text{set}}{=\!=} D$, $R(T) \stackrel{\text{set}}{=\!=} \{y \in Y \mid y = Tx,\ x \in D(T)\}$ and $N(T) \stackrel{\text{set}}{=\!=} \{x \in D(T) \mid Tx = 0\}$ are called the domain , the range and the null space of T, respectively.

If $R(T) \subset \mathbb{R}(\text{or } \mathbb{C})$, then T is said to be a linear functional on $D(T)$.

Let $T : D(T) \subset X \to Y$ be a linear operator. If $\forall x, y \in D(T),\ x \neq y$ implies $Tx \neq Ty$, then T is said to be a one-to-one map. It is easy to check that T is a one-to-one map if and only if $N(T) = \{0\}$.

Definition 1.4. Let T_1 and T_2 be linear operators with domains $D(T_1)$ and $D(T_2)$ both contained in a linear space X, and ranges $R(T_1)$ and $R(T_2)$ both contained in a linear space Y. Then $T_1 = T_2$ if and only if $D(T_1) = D(T_2)$ and $T_1 x = T_2 x,\ \forall x \in D(T_1) = D(T_2)$.

If $D(T_1) \subset D(T_2)$ and $T_1 x = T_2 x$ for all $x \in D(T_1)$, then T_2 is called an extension of T_1, and T_1 is a restriction of T_2.

Definition 1.5. A linear space X is called a normed linear space, formally written $(X, \|\cdot\|)$ (but in practice we omit $\|\cdot\|$), if for every $x \in X$, there is associated a real number $\|x\|$, the norm of x, such that

$$\begin{aligned}
&\|x\| \geq 0,\ \forall x \in X; \quad \|x\| = 0 \Longleftrightarrow x = 0,\\
&\|x+y\| \leq \|x\| + \|y\|,\ \forall x, y \in X,\\
&\|\alpha x\| = |\alpha|\|x\|, \quad \forall \alpha \in \mathbb{K},\ \forall x \in X.
\end{aligned}$$

Let X be a normed linear space. If $\lim\limits_{n\to\infty} \|x_n - x\| = 0$ for $x_n, x \in X$, then we say that the sequence $\{x_n\}$ converges strongly to x.

Definition 1.6. Let X and Y be normed linear spaces over the same scalar field $\mathbb{K}$ and $T : D(T) \subset X \to Y$ be a linear operator. If for $x \in D(T)$,

$$\lim_{n\to\infty} \|Tx_n - Tx\| = 0, \text{ for all } \{x_n\} \text{ satisfying } \lim_{n\to\infty} \|x_n - x\| = 0,$$

then T is said to be continuous at x. If T is continuous at every vector $x \in D(T)$, then T is called continuous on $D(T)$.

Definition 1.7. Let X and Y be normed linear spaces over the same scalar field $\mathbb{K}$ and $T : D(T) \subset X \to Y$ be a linear operator. If there exists a positive constant β such that

$$\|Tx\| \le \beta\|x\|, \quad \forall x \in D(T),$$

then T is called bounded on $D(T)$.

It is easy to verify that the following result holds.

Proposition 1.8. *Let X and Y be normed linear spaces over the same scalar field $\mathbb{K}$ and $T : D(T) \subset X \to Y$ be a linear operator. T is bounded if and only if T is continuous on $D(T)$.*

Definition 1.9. Let T be a continuous linear operator on a normed linear space X into a normed linear space Y. Then

$$\|T\| = \sup_{\|x\|\le 1} \|Tx\| = \sup_{\|x\|=1} \|Tx\|$$

is called the norm of T.

Definition 1.10. Let X be a normed linear space. If $\lim\limits_{\substack{n\to\infty\\ m\to\infty}} \|x_n - x_m\| = 0$ for the sequence $\{x_n\} \subset X$, then $\{x_n\}$ is called a Cauchy sequence. If every Cauchy sequence $\{x_n\}$ converges strongly to a point x_∞ of X : $\lim\limits_{n\to\infty} \|x_n - x_\infty\| = 0$, then X is called a Banach space.

Let X be a Banach space $(X, \|\cdot\|)$ endowed with an order relation $\le$ (i.e., $x \le y$ implies $x + z \le y + z$ for $x, y, z \in X$ and $x \ge 0$ implies $\lambda x \ge 0$ for $x \in X$, $\lambda \ge 0$). If the Banach space structure $(X, \|\cdot\|)$ is compatible with the order structure $(X, \le)$, that is, $|f| \le |g|$ for $f, g \in X$ implies $\|f\| \le \|g\|$, then we say that X is a Banach lattice.

Theorem 1.11 (Uniform Boundedness Theorem). *Let $\{T_\sigma \mid \sigma \in \Pi\}$ be a family of bounded linear operators defined on a Banach space X into a normed linear space Y. Then the boundedness of $\{\|T_\sigma x\| \mid \sigma \in \Pi\}$ at each $x \in X$ implies the boundedness of $\{\|T_\sigma\| \mid \sigma \in \Pi\}$.*

Theorem 1.12 (C. Neumann). *Let T be a bounded linear operator on a Banach space X into X. Suppose that $\|I - T\| < 1$, where I is the identity operator: $Ix = x$*

for all $x \in X$. Then T has a unique bounded linear inverse T^{-1} which is given by a C. Neumann series

$$T^{-1}x = \lim_{n\to\infty} \sum_{k=0}^{n} (I-T)^k x = \sum_{k=0}^{\infty} (I-T)^k x, \ \forall x \in X.$$

Theorem 1.13 (Inverse Operator Theorem). *If a continuous linear operator T on a Banach space X onto a Banach space Y gives a one-to-one map, then the inverse operator T^{-1} is also a linear continuous operator.*

Definition 1.14. Let X and Y be normed linear spaces. The graph $G(T)$ of a linear operator T on $D(T) \subset X$ into Y is the set $\{(x, Tx) \mid x \in D(T)\}$ in the product space $X \times Y$. T is called closed if and only if the following condition is satisfied: $\{x_n\} \subset D(T)$, $\lim_{n\to\infty} \|x_n - x\| = 0$ and $\lim_{n\to\infty} \|Tx_n - y\| = 0$ imply that $x \in D(T)$ and $Tx = y$.

A linear operator T on $D(T) \subset X$ into Y is said to be closable if the closure in $X \times Y$ of the graph $G(T)$ is the graph of a linear operator, say S, on $D(T) \subset X$ into Y.

Theorem 1.15 (Closed Graph Theorem). *Let X and Y be Banach spaces. If a linear operator $T : D(T) \subset X \to Y$ is closed and $D(T)$ is closed, then T is continuous.*

Theorem 1.16 (Hahn-Banach Extension Theorem). *Let X be a normed linear space, D a linear subspace of X and f_1 a continuous linear functional defined on D. Then there exists a continuous linear functional f defined on X such that*

(1) *f is an extension of f_1,*
(2) $\|f_1\| = \|f\|$.

Corollary 1.17. *Let X be a normed linear space and $x_0 \neq 0$ be any element of X. Then there exists a continuous linear functional f_0 on X such that $f_0(x_0) = \|x_0\|$ and $\|f_0\| = 1$.*

Let M be a closed linear subspace of X. Then, for any $x_0 \in X \setminus M$, there exists a continuous linear functional f_0 on X such that $f_0(x_0) > 1$ and $f_0(x) = 0$ for $x \in M$.

Definition 1.18. Let X be a normed linear space. Then X^*, the set of all continuous linear functionals on X, is called the dual space of X.

Proposition 1.19. *If X is a normed linear space, then X^* is a Banach space with the norm $|||f^*||| = \sup_{\substack{\|x\|\le 1\\ x\in X}} |f^*(x)|$ for $f^* \in X^*$.*

Theorem 1.20. $\{L^1[0,\infty)\}^* = L^\infty[0,\infty)$, *where*

$$L^1[0,\infty) = \left\{ y : [0,\infty) \to \mathbb{R} \,\middle|\, \|y\| = \int_0^\infty |y(x)|dx < \infty \right\},$$

$$L^\infty[0,\infty) = \left\{ y : [0,\infty) \to \mathbb{R} \;\middle|\; \begin{array}{l} |||y||| = \inf\limits_{mE_0=0} \sup\limits_{x\in[0,\infty)\setminus E_0} |y(x)| \\ < \infty, \quad E_0 \subset [0,\infty) \end{array} \right\}$$

are Banach spaces. Also, they are Banach lattices.

Definition 1.21. Let X and Y be normed linear spaces. Let T be a linear operator on $D(T) \subset X$ into Y. Then the linear operator T^* defined on $D(T^*) \subset Y^*$ into X^* such that

$$\langle Tx, y^*\rangle = \langle x, T^*y^*\rangle, \quad \forall x \in D(T),\ y^* \in D(T^*)$$

is called the conjugate operator or adjoint operator of T.

Definition 1.22. Let T be a linear operator whose domain $D(T)$ and range $R(T)$ both lie in the same complex normed linear space X. If γ is such that the range $R(\gamma I - T)$ is dense in X and $(\gamma I - T)$ has a continuous inverse $(\gamma I - T)^{-1}$, we say that γ is in the resolvent set $\rho(T)$ of T, and we denote this inverse $(\gamma I - T)^{-1}$ by $R(\gamma, T)$ and call it the resolvent of T. All complex numbers γ not in $\rho(T)$ form a set $\sigma(T)$ called the spectrum of T. $\sigma(T)$ is decomposed into three sets $\sigma_p(T)$, $\sigma_r(T)$, $\sigma_c(T)$ with the following properties:

1. $\sigma_p(T)$ is the totality of complex numbers γ for which $\gamma I - T$ does not have an inverse, that is, $Tx = \gamma x$ has a nonzero solution $x \in X$, which is called an eigenvector corresponding to γ. $\sigma_p(T)$ is called the point spectrum of T, and each $\gamma \in \sigma_p(T)$ is called an eigenvalue of T.
2. $\sigma_r(T)$ is the set of all $\gamma \in \mathbb{C}$ such that the range of $\gamma I - T$ is not dense in X, and so, by the Hahn-Banach extension theorem (see Theorem 1.16) $\sigma_r(T) = \sigma_p(T^*)$, the point spectrum of the adjoint operator T^* of T. $\sigma_r(T)$ is called the residual spectrum of T.
3. $\sigma_c(T)$ is the totality of complex numbers γ for which $\gamma I - T$ has a discontinuous inverse with domain dense in X. $\sigma_c(T)$ is called the continuous spectrum of T.

Definition 1.23. Let T be a linear operator whose domain $D(T)$ and range $R(T)$ both lie in the same complex normed linear space X. Assume $\gamma \in \sigma_p(T)$, then the dimension of

$$\{x \in D(T) \mid Tx = \gamma x\}$$

is called the geometric multiplicity of γ.

We call the smallest integer k such that

$$\begin{aligned} &\left\{ f \in D(T^k) \;\middle|\; (\gamma I - T)^k f = 0 \right\} \\ &= \left\{ f \in D(T^{k+1}) \;\middle|\; (\gamma I - T)^{k+1} f = 0 \right\} \end{aligned}$$

the algebraic index of γ.

The dimension of the spectral subspace

$$\mathrm{Range}\,\mathbb{P} = \mathrm{Range}\left\{ \frac{1}{2\pi i} \int_{\Gamma^*} (\gamma I - T)^{-1} d\gamma \right\}$$

is called an algebraic multiplicity of γ, where Γ^* is the circle with center γ and sufficiently small radius such that there are no other eigenvalues of T except for γ.

γ is an eigenvalue of T with algebraic multiplicity 2 if and only if $(\gamma I - T)f = g$ has a solution $f \in D(T)$ for the eigenvector g satisfying $(\gamma I - T)g = 0$.

Remark 1.24. Let T be a linear operator whose domain $D(T)$ and range $R(T)$ both lie in the same complex normed linear space X. Assume $\gamma \in \sigma_p(T)$. If the geometric multiplicity of γ and the algebraic index of γ are equal to 1 simultaneously, then the algebraic multiplicity of γ is also equal to 1.

If T is a matrix $A_{n\times n}$ and γ is an eigenvalue of $A_{n\times n}$, then the dimension (size) of Jordan block corresponding to γ is just algebraic multiplicity of γ.

Theorem 1.25. *Let T be a closed linear operator with domain and range both in a Banach space X. Then the resolvent set $\rho(T)$ is an open set of the complex plane. In each component of $\rho(T)$, $R(\gamma, T)$ is a holomorphic function of γ.*

Definition 1.26. Let X be a Banach space and $T : X \to X$ be a linear operator, then $r(T) = \sup\limits_{\gamma\in\sigma(T)} |\gamma|$ is called the spectral radius of T.

Theorem 1.27. *Let X be a Banach space and $T : X \to X$ be a bounded linear operator, then $r(T) = \lim\limits_{n\to\infty} \|T^n\|^{\frac{1}{n}}$.*

Definition 1.28. Let X be a complex Banach space and $T : X \to X$ be a bounded linear operator. Let $f(\gamma)$ be a complex-valued function which is holomorphic in some neighborhood of the spectrum $\sigma(T)$ of T and $\mathcal{U} \supset \sigma(T)$ be an open set of the complex plane which is contained in the domain of holomorphy of f. Suppose further that the boundary $\partial\mathcal{U}$ of $\mathcal{U}$ consists of a finite number of rectifiable Jordan curves, oriented in the positive sense. Then the bounded linear operator $f(T)$ will be defined by

$$f(T) = \frac{1}{2\pi i}\int_{\partial\mathcal{U}} f(\gamma)R(\gamma, T)d\gamma,$$

the integral on the right is called a Dunford integral.

Theorem 1.29 (Spectral Mapping Theorem). *Let X be a complex Banach space and $T : X \to X$ a bounded linear operator. If $f(\gamma)$ is a complex-valued function which is holomorphic in some neighborhood of the spectrum $\sigma(T)$ of T, then $f(\sigma(T)) = \sigma(f(T))$.*

Theorem 1.30. *Let γ_0 be an isolated singular point of the resolvent $R(\gamma, T)$ of a closed linear operator T on a complex Banach space X into X. Then $R(\gamma, T)$ can be expanded into a Laurent series*

$$R(\gamma, T) = \sum_{n=-\infty}^{\infty} (\gamma - \gamma_0)^n A_n, \; A_n = \frac{1}{2\pi i}\int_{\mathcal{U}} (\gamma - \gamma_0)^{-n-1} R(\gamma, T)d\gamma,$$

where $\mathcal{U}$ is a circumference of sufficiently small radius: $|\gamma - \gamma_0| = \epsilon$ such that the circle $|\gamma - \gamma_0| \leq \epsilon$ does not contain other singularities than $\gamma = \gamma_0$, and the integration is performed counter clockwise. Moreover,

$$\begin{aligned}
&TA_k y = A_k T y, \quad \forall y \in D(T), \quad k = 0, \pm 1, \pm 2, \dots, \\
&A_k A_m = 0, \quad \forall k \geq 0, \quad m \leq -1, \\
&A_n = (-1)^n A_0^{n+1}, \quad n \geq 1, \\
&A_{-n-m+1} = A_{-m} A_{-n}, \quad n \geq 1, \quad m \geq 1.
\end{aligned}$$

Definition 1.31. Let X be a normed linear space and $Y \subset X$. If any series in Y has a convergent subseries, then Y is called a relatively compact set. Furthermore, Y is said to be a compact set if the limit of the convergent subseries belongs to Y.

Definition 1.32. Let X be a normed linear space, Y_1 be a subset of X, Y_2 be a subset of Y_1. If there is a $\epsilon > 0$ such that

$$Y_1 \subset \bigcup_{y \in Y_2} O(y, \epsilon),$$

then Y_2 is called the ϵ-net of Y_1. Here $O(y, \epsilon)$ is the ϵ-neighborhood of y.

If for any $\epsilon > 0$, Y_1 has a finite ϵ-net, then Y_1 is called a totally bounded set.

Theorem 1.33 (Hausdorff). *A subset in a Banach space is relatively compact if and only if it is totally bounded.*

Theorem 1.34 (Fréchet-Kolmogorov). *$Y \subset L^1[0, \infty)$ is relatively compact if and only if it satisfies the following conditions:*

1. $\sup_{y \in Y} \|y\| = \sup_{y \in Y} \int_0^\infty |y(x)| dx < \infty.$
2. $\lim_{h \to 0} \int_0^\infty |y(x+h) - y(x)| dx = 0$ *uniformly for* $y \in Y$.
3. $\lim_{h \to \infty} \int_h^\infty |y(x)| dx = 0$ *uniformly for* $y \in Y$.

From the above two theorems we deduce the following result which will be used in later chapters.

Theorem 1.35 (Gupur [57]). *A subset Y of*

$$X = \left\{ y \;\middle|\; \begin{array}{l} y \in \mathbb{R} \times L^1[0, \infty) \times L^1[0, \infty) \times \cdots \\ \|y\| = |y_0| + \sum\limits_{n=1}^{\infty} \|y_n\|_{L^1[0,\infty)} < \infty \end{array} \right\}$$

is relatively compact if and only if the following conditions hold simultaneously:

1. $\sup\limits_{y \in Y} \|y\| < \infty$.
2. $\lim\limits_{h\to 0} \sum\limits_{n=1}^{\infty} \int_0^\infty |y_n(x+h) - y_n(x)|\mathrm{d}x = 0,$ *uniformly for*
$$y = (y_0, y_1, y_2, \dots) \in Y.$$
3. $\lim\limits_{h\to\infty} \sum\limits_{n=1}^{\infty} \int_h^\infty |y_n(x)|\mathrm{d}x = 0,$ *uniformly for*
$$y = (y_0, y_1, y_2, \dots) \in Y.$$
4. *For* $\forall \epsilon > 0$ *there exists natural number* $N(\epsilon)$ *which only depends on* ϵ *such that*
$$\sum_{n=N(\epsilon)+1}^{\infty} \int_0^\infty |y_n(x)|dx < \epsilon$$
for all $y = (y_0, y_1, y_2, \dots) \in Y$.

Proof. Necessity. Since Y is relatively compact, from the Hausdorff theorem (see Theorem 1.33) Y is totally bounded and therefore Y is bounded, i.e.,

$$\sup_{y\in Y} \|y\| < \infty. \tag{1-1}$$

Since Y is totally bounded, for $\forall \epsilon > 0$ take a finite $\frac{\epsilon}{2}$-net of Y,

$$y^{(1)}, y^{(2)}, \dots, y^{(m)}, \quad \text{where}$$
$$y^{(i)} = (y_1^{(i)}, y_2^{(i)}, y_3^{(i)}, \dots), \quad i = 1, 2, \dots, m,$$

then, for $\forall y \in Y$, there exists $\exists\, y^{(j)}$ such that

$$\sum_{n=1}^{\infty} \left\| y_n - y_n^{(j)} \right\| = \left\| y - y^{(j)} \right\| < \frac{\epsilon}{2}. \tag{1-2}$$

Because $y^{(i)} \in Y \subset X$,

$$\left\| y^{(i)} \right\| < \infty \Rightarrow \forall \epsilon > 0,\ \exists\, N_i(\epsilon) \text{ such that}$$
$$\sum_{n=N_i(\epsilon)+1}^{\infty} \left\| y_n^{(i)} \right\|_{L^1[0,\infty)} < \frac{\epsilon}{2},\ 1 \le i \le m. \tag{1-3}$$

If we take $N(\epsilon) = \max\{N_1(\epsilon), N_2(\epsilon), \dots, N_m(\epsilon)\}$, then (1-3) gives

$$\sum_{n=N(\epsilon)+1}^{\infty} \left\| y_n^{(i)} \right\|_{L^1[0,\infty)} < \frac{\epsilon}{2}, \quad 1 \le i \le m. \tag{1-4}$$

By combining (1-2) with (1-4) we have

$$\begin{aligned}\sum_{n=N(\epsilon)+1}^{\infty} \|y_n\|_{L^1[0,\infty)} &\leq \sum_{n=N(\epsilon)+1}^{\infty} \left\|y_n - y_n^{(j)}\right\|_{L^1[0,\infty)} + \sum_{n=N(\epsilon)+1}^{\infty} \left\|y_n^{(j)}\right\|_{L^1[0,\infty)} \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \end{aligned} \tag{1-5}$$

From relative compactness of Y we know that

$$Y_n = \{y_n \mid y = (y_0, y_1, y_2, \dots) \in Y\} \subset L^1[0,\infty),\ n = 1, 2, \dots,$$

are relatively compact. Hence, by the Fréchet-Kolmogorov theorem (see Theorem 1.34) it follows that

$$\lim_{h\to 0} \int_0^{\infty} |y_n(x+h) - y_n(x)|dx = 0, \text{ uniformly for } y_n \in Y_n. \tag{1-6}$$

$$\lim_{h\to \infty} \int_h^{\infty} |y_n(x)|dx = 0, \quad \text{uniformly for } y_n \in Y_n. \tag{1-7}$$

(1-6) gives

$$\begin{aligned}&\lim_{h\to 0} \sum_{n=1}^{N(\epsilon)} \int_0^{\infty} |y_n(x+h) - y_n(x)|dx \\ &= \sum_{n=1}^{N(\epsilon)} \lim_{h\to 0} \int_0^{\infty} |y_n(x+h) - y_n(x)|dx = 0 \\ &\Longrightarrow \\ &\forall \epsilon > 0,\ \exists \delta(\epsilon) > 0,\ \forall h : |h| < \delta(\epsilon) \\ &\sum_{n=1}^{N(\epsilon)} \int_0^{\infty} |y_n(x+h) - y_n(x)|dx < \epsilon. \end{aligned} \tag{1-8}$$

(1-7) implies

$$\begin{aligned}&\lim_{h\to \infty} \sum_{n=1}^{N(\epsilon)} \int_h^{\infty} |y_n(x)|dx \\ &= \sum_{n=1}^{N(\epsilon)} \lim_{h\to \infty} \int_h^{\infty} |y_n(x)|dx = 0 \\ &\Longrightarrow \\ &\forall \epsilon > 0,\ \exists \mathcal{K}(\epsilon) > 0,\ \forall h : |h| > \mathcal{K}(\epsilon) \\ &\sum_{n=1}^{N(\epsilon)} \int_h^{\infty} |y_n(x)|dx < \epsilon. \end{aligned} \tag{1-9}$$

(1-5) and (1-8) imply: $\forall \epsilon > 0,\ \exists N(\epsilon),\ \delta(\epsilon) > 0,\ \forall h : |h| < \delta(\epsilon)$,

$$\begin{aligned}
&\sum_{n=1}^{\infty} \int_0^{\infty} |y_n(x+h) - y_n(x)|dx \\
&= \sum_{n=1}^{N(\epsilon)} \int_0^{\infty} |y_n(x+h) - y_n(x)|dx \\
&\quad + \sum_{n=N(\epsilon)+1}^{\infty} \int_0^{\infty} |y_n(x+h) - y_n(x)|dx \\
&\le \epsilon + \sum_{n=N(\epsilon)+1}^{\infty} \|y_n\|_{L^1[0,\infty)} + \sum_{n=N(\epsilon)+1}^{\infty} \|y_n\|_{L^1[0,\infty)} \\
&< \epsilon + \epsilon + \epsilon = 3\epsilon.
\end{aligned} \tag{1-10}$$

From (1-5) together with (1-9) we obtain: $\forall \epsilon > 0,\ \exists N(\epsilon),\ \mathcal{K}(\epsilon),\ \forall h : |h| > \mathcal{K}(\epsilon)$,

$$\begin{aligned}
\sum_{n=1}^{\infty} \int_h^{\infty} |y_n(x)|dx &= \sum_{n=1}^{N(\epsilon)} \int_h^{\infty} |y_n(x)|dx + \sum_{n=N(\epsilon)+1}^{\infty} \int_h^{\infty} |y_n(x)|dx \\
&\le \sum_{n=1}^{N(\epsilon)} \int_h^{\infty} |y_n(x)|dx + \sum_{n=N(\epsilon)+1}^{\infty} \int_0^{\infty} |y_n(x)|dx \\
&< \epsilon + \epsilon = 2\epsilon.
\end{aligned} \tag{1-11}$$

(1-1), (1-5), (1-10) and (1-11) show that the four conditions of this theorem hold.

Sufficiency. For all $\eta > 0$ take $\epsilon = \frac{\eta}{2} > 0$. From condition 4 we know that for $\epsilon > 0$ there is an $N(\epsilon)$ such that

$$\sum_{n=N(\epsilon)+1}^{\infty} \int_0^{\infty} |y_n(x)|dx < \epsilon. \tag{1-12}$$

By using condition 2, condition 3 and the relation

$$\begin{aligned}
&\int_0^{\infty} |y_n(x+h) - y_n(x)|dx \\
&\le \sum_{n=1}^{\infty} \int_0^{\infty} |y_n(x+h) - y_n(x)|dx, \forall n \ge 1,
\end{aligned} \tag{1-13}$$

$$\int_h^{\infty} |y_n(x)|dx \le \sum_{n=1}^{\infty} \int_h^{\infty} |y_n(x)|dx,\ \forall n \ge 1, \tag{1-14}$$

we have

$$\lim_{h \to 0} \int_0^{\infty} |y_n(x+h) - y_n(x)|dx = 0, \text{ uniformly for } y_n, \tag{1-15}$$

$$\lim_{h\to\infty}\int_h^\infty |y_n(x)|dx = 0, \text{ uniformly for } y_n. \tag{1-16}$$

From (1-12), (1-15), (1-16) and the Fréchet-Kolmogorov theorem we obtain that $Y_n = \{y_n \mid (y_0, y_1, y_2, \dots) \in Y\} \subset L^1[0,\infty)$ ($\forall n \geq 0$) are relatively compact. If we define

$$Z = \left\{(y_0, y_1, \dots, y_{N(\epsilon)}) \mid (y_0, y_1, y_2, \dots) \in Y\right\},$$

then we will prove that Z is relatively compact.

Take any series $\left(y_0^{(k)}, y_1^{(k)}, y_2^{(k)}, \dots, y_{N(\epsilon)}^{(k)}\right) \in Z$, then condition 1 ensures this series is bounded, i.e.,

$$\left|y_0^{(k)}\right| + \sum_{n=1}^{N(\epsilon)} \left\|y_n^{(k)}\right\|_{L^1[0,\infty)} < \infty \Rightarrow \left|y_0^{(k)}\right| < \infty, \ \forall k. \tag{1-17}$$

From which, together with $y_0^{(k)} \in \mathbb{R}$, we know that $\left\{y_0^{(k)} \mid k \geq 1\right\} \subset \mathbb{R}$ is relatively compact. So, there exists a convergence subseries $y_0^{k_0}$ such that

$$\lim_{k_0\to\infty} y_0^{(k_0)} = y_0 \in \mathbb{R}. \tag{1-18}$$

By $y_0^{(k_0)}$ we can select corresponding subseries $y_1^{(k_0)}, y_2^{(k_0)}, \dots, y_{N(\epsilon)}^{(k_0)}$ of $y_1^{(k)}, y_2^{(k)}$, $\dots, y_{N(\epsilon)}^{(k)}$. Since Y_1 is relatively compact, there is a subseries $y_1^{(k_1)}$ of $y_1^{(k_0)}$ such that

$$\lim_{k_1\to\infty} y_1^{(k_1)} = y_1 \in L^1[0,\infty). \tag{1-19}$$

Through $y_1^{(k_1)}$ we can select corresponding subseries $y_2^{(k_1)}, \dots, y_{N(\epsilon)}^{(k_1)}$ of $y_2^{(k_0)}$, $\dots, y_{N(\epsilon)}^{(k_0)}$. Because Y_2 is relatively compact, there exists a subseries $y_2^{(k_2)}$ of $y_2^{(k_1)}$ such that

$$\lim_{k_2\to\infty} y_2^{(k_2)} = y_2 \in L^1[0,\infty). \tag{1-20}$$

According to $y_2^{(k_2)}$ we select corresponding subseries $y_3^{(k_2)}, y_4^{(k_2)}, \dots, y_{N(\epsilon)}^{(k_2)}$ of $y_3^{(k_1)}, y_4^{(k_1)}, \dots, y_{N(\epsilon)}^{(k_1)}$. From relative compactness of Y_3 we derive that there is a subseries $y_3^{(k_3)}$ of $y_3^{(k_2)}$ such that

$$\lim_{k_3\to\infty} y_3^{(k_3)} = y_3 \in L^1[0,\infty). \tag{1-21}$$

By $y_3^{(k_3)}$ we choose corresponding subseries of $y_4^{(k_3)}, y_5^{(k_3)}, \dots, y_{N(\epsilon)}^{(k_3)}$ of $y_4^{(k_2)}$, $y_5^{(k_2)}, \dots, y_{N(\epsilon)}^{(k_2)}$. Through repeating the above process we obtain a convergent subseries $y_i^{(k_i)}$ such that

$$\lim_{k_i\to\infty} y_i^{(k_i)} = y_i \in L^1[0,\infty), \ i = 0, 1, 2, \dots N(\epsilon). \tag{1-22}$$

From the construction of $y_i^{(k_i)}$ it is easy to see that $y_i^{(k_i)}$ is a subseries of $y_i^{(k_{i-1})}$, $y_i^{(k_{i-2})}$ is a subseries of $y_i^{(k_{i-1})}$, $\ldots$, $y_i^{(k_1)}$ is a subseries of $y_i^{(k_0)}$. Because all subseries of a convergent series converge to the same limit, from (1-22) we have

$$\lim_{k_{N(\epsilon)}\to\infty} y_i^{(k_{N(\epsilon)})} = y_i \in L^1[0,\infty),\ i = 0,1,2,\ldots,N(\epsilon). \tag{1-23}$$

By (1-23) it is immediately deduced that

$$\begin{aligned}
&\lim_{k_{N(\epsilon)}\to\infty} \left\{ \left| y_0^{(k_{N(\epsilon)})} - y_0 \right| + \sum_{i=1}^{N(\epsilon)} \int_0^\infty \left| y_i^{(k_{N(\epsilon)})}(x) - y_i(x) \right| dx \right\} \\
&= \lim_{k_{N(\epsilon)}\to\infty} \left| y^{(k_{N(\epsilon)})} - y_0 \right| \\
&\quad + \lim_{k_{N(\epsilon)}\to\infty} \sum_{i=1}^{N(\epsilon)} \int_0^\infty \left| y_i^{(k_{N(\epsilon)})}(x) - y_i(x) \right| dx \\
&= \sum_{i=1}^{N(\epsilon)} \lim_{k_{N(\epsilon)}\to\infty} \int_0^\infty \left| y_i^{(k_{N(\epsilon)})}(x) - y_i(x) \right| dx \\
&= 0.
\end{aligned} \tag{1-24}$$

(1-24) shows that any series in Z has a convergent subseries and therefore Z is relatively compact. The Fréchet-Kolmogorov theorem (see Theorem 1.34) ensures that Z is totally bounded. Take an ϵ-net of Z, then for $\forall y = (y_0, y_1, y_2, \ldots) \in Y$ there is $y^{(k)} = (y_0^{(k)}, y_1^{(k)}, y_2^{(k)}, \ldots, y_{N(\epsilon)}^{(k)}) \in Z,\ k = 1,2,\ldots,l$ such that

$$\left| y_0^{(k)} - y_0 \right| + \sum_{n=1}^{N(\epsilon)} \int_0^\infty \left| y_n^{(k)}(x) - y_n(x) \right| dx < \epsilon. \tag{1-25}$$

From (1-12) and (1-25) it follows that

$$\begin{aligned}
\left\| y^{(k)} - y \right\| &= \left| y_0^{(k)} - y_0 \right| + \sum_{n=1}^{\infty} \int_0^\infty \left| y_n^{(k)}(x) - y_n(x) \right| dx \\
&= \left| y_0^{(k)} - y_0 \right| + \sum_{n=1}^{N(\epsilon)} \int_0^\infty \left| y_n^{(k)}(x) - y_n(x) \right| dx \\
&\quad + \sum_{n=N(\epsilon)+1}^{\infty} \int_0^\infty |y_n(x)| dx \\
&< \epsilon + \epsilon \overset{\text{set}}{=\!=} \eta, \quad \forall y \in Y,
\end{aligned} \tag{1-26}$$

where we regard $y^{(k)}$ as

$$y^{(k)} = (y_0^{(k)}, y_1^{(k)}, y_2^{(k)}, \ldots, y_{N(\epsilon)}^{(k)}, 0, 0, \ldots).$$

(1-26) shows that Z is an η-net of Y and therefore Y is totally bounded. Since X is a Banach space, from the Hausdorff theorem (see Theorem 1.33) we conclude that Y is relatively compact. □

From Theorem 1.35 and its proof we deduce the following corollaries:

Corollary 1.36. *A subset Y of*

$$X = \left\{ y \;\middle|\; \begin{array}{l} y \in \mathbb{R}^k \times L^1[0,\infty) \times L^1[0,\infty) \times \cdots, \\ \|y\| = \sum\limits_{n=0}^{k-1} |y_{n,0}| + \sum\limits_{n=0}^{\infty} \|y_{n,1}\|_{L^1[0,\infty)} < \infty \end{array} \right\}$$

is relatively compact if and only if it satisfies:

1. $\sup_{y \in Y} \|y\| < \infty$,
2. $\lim\limits_{h\to 0} \sum\limits_{n=0}^{\infty} \int_0^\infty |y_{n,1}(x+h) - y_{n,1}(x)| \mathrm{d}t = 0$, *uniformly for*
$$y = (y_{0,0}, y_{0,1}, \ldots, y_{k-1,0}, y_{0,1}, y_{1,1}, y_{2,1}, \ldots) \in Y,$$
3. $\lim\limits_{h\to \infty} \sum\limits_{n=0}^{\infty} \int_h^\infty |y_{n,1}(x)| \mathrm{d}x = 0$, *uniformly for*
$$y = (y_{0,0}, y_{0,1}, \ldots, y_{k-1,0}, y_{0,1}, y_{1,1}, y_{2,1}, \ldots) \in Y.$$
4. *For* $\forall \epsilon > 0$ *there exists a natural number* $N(\epsilon)$ *which depends only on* ϵ *such that*
$$\sum_{n=N(\epsilon)+1}^{\infty} \int_0^\infty |y_{n,1}(x)| dx < \epsilon$$
for all $y = (y_{0,0}, y_{0,1}, \ldots, y_{k-1,0}, y_{0,1}, y_{1,1}, y_{2,1}, \ldots) \in Y$.

Corollary 1.37. *A subset Y of*

$$X = \left\{ y \;\middle|\; \begin{array}{l} y \in \mathbb{R} \times L^1[0,\infty) \times L^1[0,\infty), \\ \|y\| = |y_0| + \sum\limits_{n=1}^{2} \|y_n\|_{L^1[0,\infty)} \end{array} \right\}$$

is relatively compact if and only if the following conditions satisfy simultaneously:

1. $\lim\limits_{h\to 0} \sum\limits_{n=1}^{2} \int_0^\infty |y_n(x+h) - y_n(x)| dx = 0$, *uniformly for* $y \in Y$,
2. $\lim\limits_{h\to \infty} \sum\limits_{n=1}^{2} \int_h^\infty |y_n(x)| dx = 0$, *uniformly for* $y \in Y$.

Definition 1.38. Let X and Y be Banach spaces. A linear operator $T : X \to Y$ is said to be compact if T maps every bounded subset of X into a relatively compact subset of Y.

T is called a Fredholm operator if there exists a linear bounded operator $\mathcal{R} : X \to X$ such that $I - \mathcal{R}T$ and $I - T\mathcal{R}$ are compact.

Theorem 1.39.

1. *The unit ball of* X^* *is compact in the weak star topology, that is,* $\forall\{q_n^* | \ n \geq 1\} \subset \{f^* \in X^* \mid |||f^*||| \leq 1\}$ *has a cluster point* q^* *such that* $\lim\limits_{n\to\infty} \langle q_n^*, f\rangle = \langle q^*, f\rangle$ *for all* $f \in X$.
2. *Any compact operator is bounded.*
3. *A linear combination of compact operators is compact.*
4. *The product of a compact operator with a bounded linear operator is compact.*
5. *Let a sequence* $\{T_n\}$ *of compact operators converge to an operator* T *in the sense of* $\lim\limits_{n\to\infty} \|T_n - T\| = 0$, *then* T *is compact.*

Theorem 1.40. *Let* X *be a Banach space and* $T : X \to X$ *be compact. Then*

1. *the spectrum of* T *consists of an at most countable set of points of the complex plane which has no point of accumulation except possibly* 0;
2. *every nonzero number in the spectrum of* T *is an eigenvalue of* T *with finite algebraic multiplicity;*
3. *a nonzero number is an eigenvalue of* T *if and only if it is an eigenvalue of* T^*.

Definition 1.41. Let X be a complex Banach space, and let I be an interval (bounded or unbounded) in $\mathbb{R}$, or a rectangle in $\mathbb{R}^2$. A function $f : I \to X$ is said to be simple if it is of the form $f(t) \xlongequal{\text{set}} \sum\limits_{k=1}^{n} y_k \chi_{\Omega_k}(t)$ for some $n \in \mathbb{N} = \{1, 2, \dots\}$, $y_k \in X$ and Lebesgue measurable sets $\Omega_k \subset I$ with finite Lebesgue measure $m(\Omega_k)$; f is said to be a step function when each Ω_k can be chosen to be an interval, or a rectangle in $\mathbb{R}^2$. Here χ_Ω denotes the characteristic function of Ω.

A function $f : I \to X$ is called measurable if there is a sequence of simple functions g_n such that $f(t) = \lim\limits_{n\to\infty} g_n(t)$ for almost all $t \in I$.

If there is a countable partition $\{\Omega_n \mid n \in \mathbb{N}\}$ of I into subsets Ω_n such that f is constant on each Ω_n, then we say that $f : I \to X$ is countable valued.

$f : I \to X$ is called almost separably valued if there is a null set Ω_0 in I such that $f(I \setminus \Omega_0) \xlongequal{\text{set}} \{f(t) \mid t \in I \setminus \Omega_0\}$ is separable. $f : I \to X$ is called weakly measurable if $y^* \circ f : t \to \langle f(t), y^*\rangle$ is Lebesgue measurable for each y^* in the dual space X^* of X.

Theorem 1.42. *Let* X *be a complex Banach space,* I *be an interval* (*bounded or unbounded*) *in* $\mathbb{R}$, *or a rectangle in* $\mathbb{R}^2$ *and* $f : I \to X$. *Then the following statements hold:*

1. f *is measurable if and only if it is the uniform limit almost everywhere of a sequence of measurable, countably-valued functions.*
2. *If* X *is separable, then* f *is measurable if and only if it is weakly measurable.*
3. *If* f *is continuous, then it is measurable.*
4. *If* $f_n : I \to X$ *are measurable functions and* $f_n \to f$ *pointwise almost everywhere, then* f *is measurable.*

For a simple function $g : I \to X$, $g = \sum\limits_{i=1}^{n} y_i \chi_{\Omega_i}$, we define

$$\int_I g(t)dt \xlongequal{\text{set}} \sum_{i=1}^{n} y_i m(\Omega_i),$$

where $m(\Omega_i)$ is the Lebesgue measure of $\Omega_i,\ i = 1, 2, \ldots, n$.

Definition 1.43. Let X be a complex Banach space, and let I be an interval (bounded or unbounded) in $\mathbb{R}$, or a rectangle in $\mathbb{R}^2$. A function $f : I \to X$ is called Bochner integrable if there exist simple functions g_n such that $g_n \to f$ pointwise almost everywhere and $\lim\limits_{n\to\infty} \int_I \|g_n(t) - f(t)\|dt = 0$. If $f : I \to X$ is Bochner integrable, then the Bochner integral of f on I is

$$\int_I f(t)dt = \lim_{n\to\infty} \int_I g_n(t)dt.$$

Theorem 1.44 (Bochner). *Let X be a complex Banach space, and let I be an interval* (*bounded or unbounded*) *in $\mathbb{R}$, or a rectangle in $\mathbb{R}^2$. A function $f : I \to X$ is Bochner integrable if and only if f is measurable and $\|f\|$ is integrable. If f is Bochner integrable, then*

$$\left\| \int_I f(t)dt \right\| \le \int_I \|f(t)\|dt.$$

Theorem 1.45. *Let I be an interval in $\mathbb{R}$, or a rectangle in $\mathbb{R}^2$. Let $T : X \to Y$ be a bounded linear operator between Banach spaces X and Y, and let $f : I \to X$ be Bochner integrable. Then $T \circ f : t \to T(f(t))$ is Bochner integrable and*

$$T \int_I f(t)dt = \int_I T(f(t))dt.$$

Theorem 1.46. *Let I be an interval in $\mathbb{R}$, or a rectangle in $\mathbb{R}^2$. Let A be a closed linear operator on X. Let $f : I \to X$ be Bochner integrable. Suppose that $f(t) \in D(A)$ for all $t \in I$ and $A \circ f : I \to X$ is Bochner integrable. Then $\int_I f(t)dt \in D(A)$ and*

$$A \int_I f(t)dt = \int_I A(f(t))dt.$$

Theorem 1.47 (Dominated Convergence Theorem). *Let I be an interval in $\mathbb{R}$, or a rectangle in $\mathbb{R}^2$ and X be a Banach space. Let $f_n : I \to X$ be Bochner-integrable functions. If $f(t) \xlongequal{\text{set}} \lim\limits_{n\to\infty} f_n(t)$ exists almost everywhere and if there exists an integrable function $g : I \to \mathbb{R}$ such that $\|f_n(t)\| \le g(t)$ almost everywhere for all $n \in \mathbb{N}$, then f is Bochner integrable and $\int_I f(t)dt = \lim\limits_{n\to\infty} \int_I f_n(t)dt$. Moreover, $\lim\limits_{n\to\infty} \int_I \|f_n(t) - f(t)\|dt = 0$.*

Definition 1.48. Let X be a Banach space and $f : [0, \infty) \to X$. If the limit $\lim\limits_{\tau\to\infty} \int_0^\tau f(t)dt$ exists, then we define

$$\int_0^\infty f(t)dt = \lim_{\tau\to\infty} \int_0^\tau f(t)dt.$$

If $\int_0^\infty \|f(t)\|dt < \infty$, then we say that the integral is absolutely convergent.

Theorem 1.49 (Fubini). *Let $I = I_1 \times I_2$ be a rectangle in $\mathbb{R}^2$ and $f : I \to X$ be measurable, and suppose*

$$\int_{I_1}\int_{I_2} \|f(s,t)\|dtds < \infty.$$

Then f is Bochner integrable and the repeated integrals

$$\int_{I_1}\int_{I_2} f(s,t)dtds, \quad \int_{I_2}\int_{I_1} f(s,t)dsdt$$

exist and are equal, and they coincide with the double integral $\int_I f(s,t)d(s,t)$, i.e.,

$$\int_I f(s,t)d(s,t) = \int_{I_1}\int_{I_2} f(s,t)dtds = \int_{I_2}\int_{I_1} f(s,t)dsdt.$$

Definition 1.50. Let X be a Banach space. If for every $\epsilon > 0$ there exists $\delta > 0$ such that $\sum\limits_i \|F(b_i) - F(a_i)\| < \epsilon$ for every finite collection $\{(a_i, b_i)\}$ of disjoint intervals in $[a, b]$ with $\sum\limits_i (b_i - a_i) < \delta$, then we say that F is absolutely continuous on $[a, b]$.

We say that F is Lipschitz continuous on $[a, b]$ if there exists $\beta > 0$ such that $\|F(t) - F(s)\| \leq \beta|t - s|$ for all $s, t \in [a, b]$.

Proposition 1.51. *Let X be a Banach space, $f : [a, b] \to X$ be Bochner integrable and $F(t) \xlongequal{\text{set}} \int_a^t f(s)ds$, $\forall t \in [a, b]$. Then:*

1. *F is differentiable almost everywhere and $F' = f$ almost everywhere.*
2. $\lim\limits_{h\to 0} \int_t^{t+h} \|f(s) - f(t)\|ds = 0$ *for almost all t.*
3. *F is absolutely continuous.*

Proposition 1.52. *Let X be a Banach space. Let $F : [a, b] \to X$ be absolutely continuous and $f(t) \xlongequal{\text{set}} F'(t)$. Then f is Bochner integrable and $F(t) = F(a) + \int_a^t f(s)ds$ for all $t \in [a, b]$.*

Definition 1.53. Let X be a complex Banach space and

$$L^1_{\text{loc}}(\mathbb{R}_+, X) \xlongequal{\text{set}} \left\{ f : \mathbb{R}_+ \to X \;\middle|\; \begin{array}{l} f \text{ is Bochner integrable} \\ \text{on } [0, \tau] \text{ for all } \tau > 0 \end{array} \right\}.$$

Here $\mathbb{R}_+ = [0, \infty)$. For $f \in L^1_{loc}(\mathbb{R}_+, X)$ and $\gamma \in \mathbb{C}$, the integral

$$\hat{f}(\gamma) \stackrel{\text{set}}{=\!=} \int_0^\infty e^{-\gamma t} f(t)dt \stackrel{\text{set}}{=\!=} \lim_{\tau\to\infty} \int_0^\tau e^{-\gamma t} f(t)dt$$

is called the Laplace integral.

We call $\operatorname{abs}(f) \stackrel{\text{set}}{=\!=} \{\Re\gamma \mid \hat{f}(\gamma) \text{ exists}\}$ the abscissa of convergence of $\hat{f}$ and call

$$\omega(f) \stackrel{\text{set}}{=\!=} \left\{\omega \in \mathbb{R} \;\middle|\; \sup_{t\geq 0} \left\|e^{-\omega t} f(t)\right\| < \infty\right\}$$

the exponential growth bound of f.

Theorem 1.54. *Let X be a complex Banach space and $f \in L^1_{\text{loc}}(\mathbb{R}_+, X)$, $F(t) \stackrel{\text{set}}{=\!=} \int_0^t f(s)ds$,*

$$F_\infty = \begin{cases} \lim_{t\to\infty} F(t) & \lim_{t\to\infty} F(t) \textit{ exists}, \\ 0 & \textit{otherwise}; \end{cases}$$

then $abs(f) = \omega(F - F_\infty)$.

Theorem 1.55. *Let X and Y be Banach spaces and $T : \mathbb{R}_+ \to \mathcal{L}(X, Y)$ ($\mathcal{L}(X, Y)$ is the set consisting of all bounded linear operators from X to Y) be strongly continuous, let $S(t) = \int_0^t T(s)ds$,*

$$S_\infty = \begin{cases} \lim_{t\to\infty} S(t) & \lim_{t\to\infty} S(t) \textit{ exists}, \\ 0 & \textit{otherwise}; \end{cases}$$

then

1. $\lim\limits_{t\to\infty} \int_0^t e^{-\gamma s} T(s)ds$ *exists in operator norm whenever* $\Re\gamma > \operatorname{abs}(T)$. ($\Re\gamma$ *means the real part of* γ.)
2. $$\operatorname{abs}(T) \stackrel{\text{set}}{=\!=} \inf \left\{ \Re\gamma \;\middle|\; \begin{array}{l} \int_0^t e^{-\gamma s} T(s)ds \textit{ converges} \\ \textit{strongly as } t \to \infty \end{array} \right\}$$

   are equal to

   $$\operatorname{abs}(T) = \inf \left\{ \gamma \in \mathbb{R} \;\middle|\; \begin{array}{l} \sup\limits_{t\geq 0} \left|\int_0^t e^{-\gamma s} \langle T(s)y, y^*\rangle\right| < \infty \\ \forall y \in X, \quad \forall y^* \in X^* \end{array} \right\}.$$

3. $\operatorname{abs}(T) = \omega(S - S_\infty)$.

Theorem 1.56. *Let X be a complex Banach space and $f \in L^1_{\text{loc}}(\mathbb{R}_+, X)$ with $\operatorname{abs}(f) < \infty$. Then $\gamma \to \hat{f}(\gamma)$ is holomorphic for $\Re\gamma > \operatorname{abs}(f)$ and $f^{(n)}(\gamma) = \int_0^\infty e^{-\gamma t}(-t)^n f(t)dt$ as an improper Bochner integral for $n \geq 0$, $\Re\gamma > \operatorname{abs}(f)$.*

1.2 Well-posedness of an Abstract Cauchy Problem

The well-posedness of an abstract Cauchy problem is related directly to a C_0-semigroup of linear operators. Hence, we will introduce the theory of C_0-semigroups of linear operators. In this part we refer to Pazy [95], Gupur et al. [65], Phillips [96], Arendt et al. [6] and Engel et al. [27].

Let X be a Banach space and $\mathcal{L}(X) = \mathcal{L}(X, X)$ be the set of all bounded linear operators T from X to X itself.

Definition 1.57. A C_0-semigroup (or strongly continuous semigroup) of linear operators on a Banach space X is a family $\{T(t)\}_{t\geq 0} \subset \mathcal{L}(X)$ (for short, write $T(t)$) such that

(i) $T(s)T(t) = T(s+t), \quad \forall t, s \in [0, \infty)$.
(ii) $T(0) = I$, the identity operator on X.
(iii) $\lim\limits_{t\to 0+} \|T(t)f - f\| = 0$, for each fixed $f \in X$.

Remark 1.58.

(i) The restriction that each operator $T(t)$ is in $\mathcal{L}(X)$ allows us to control errors in the measurement of the initial state u_0, because
$$\|T(t)u_0 - T(t)v_0\| \leq \|T(t)\|\|u_0 - v_0\|.$$
The continuity condition (iii) turns out to be enough to ensure that there is no disastrous breakdown in the system as time goes on.
(ii) Condition (i) in Definition 1.57 justifies the word "semigroup" to describe the algebraic structure. Indeed we have a commutative (or Abelian) semigroup with identity element, namely $T(0)$.
(iii) Most of the results that follow still hold if the domain of $T(t)$ is a suitable closed subset $D_0 \subset X$.
(iv) If A is a bounded linear operator, then it is easy to verify that the power series defined by $T(t) = e^{tA} = \sum\limits_{n=0}^{\infty} \frac{(tA)^n}{n!}$ satisfies all conditions in Definition 1.57, i.e., $T(t)$ is a C_0-semigroup.

Lemma 1.59. *Let $T(t)$ be a C_0-semigroup. There exist constants $\omega \geq 0$ and $\mathcal{B} \geq 1$ such that*
$$\|T(t)\| \leq \mathcal{B}e^{\omega t}, \quad \forall t \in [0, \infty).$$

Proof. We show first that there is an $\eta > 0$ such that $\|T(t)\|$ is bounded for $0 \leq t \leq \eta$. If this is false, then there is a sequence $\{t_n\}$ satisfying $t_n \geq 0$, $\lim\limits_{n\to\infty} t_n = 0$ and $\|T(t)\| \geq n$. From the uniform boundedness theorem (see Theorem 1.11) it then follows that for some $f \in X$, $\|T(t)f\|$ is unbounded which is contrary to continuity of $T(t)$ at $t = 0$. Thus, there exists a positive constant $\mathcal{B}$ such that $\|T(t)\| \leq \mathcal{B}$ for $0 \leq t \leq \eta$. Since $\|T(0)\| = 1$, $\mathcal{B} \geq 1$. Let $\omega = \eta^{-1} \ln \mathcal{B} \geq 0$.

Given $t > \eta$, then there exist a positive integer n and a positive number $\delta \in [0, \eta)$ such that $t = n\eta + \delta$ and therefore by the semigroup property

$$\begin{aligned}\|T(t)\| &= \|T(n\eta + \delta)\| = \|T(\delta)T(n\eta)\| \\ &\le \|T(\delta)\|\|T(n\eta)\| \\ &= \|T(\delta)\|\|T(\overbrace{\eta + \eta + \cdots + \eta}^{n})\| \\ &= \|T(\delta)\|\|\overbrace{T(\eta)T(\eta)\cdots T(\eta)}^{n}\| \\ &= \|T(\delta)\|\|(T(\eta))^n\| \\ &\le \mathcal{B} \times \mathcal{B}^n = \mathcal{B}(e^{\omega\eta})^n \\ &= \mathcal{B}e^{\omega n\eta} = \mathcal{B}e^{\omega(t-\delta)} \\ &\le \mathcal{B}e^{\omega t}.\end{aligned}$$

From which together with $\|T(t)\| \le \mathcal{B} \le \mathcal{B}e^{\omega t}$ for $t \in [0, \eta]$ we know that the result of this lemma is right. □

If $\omega = 0$ and $\mathcal{B} = 1$ in Lemma 1.59, then we call $T(t)$ a contraction C_0-semigroup.

Lemma 1.60. *For any C_0-semigroup $\{T(t)\}_{t\ge 0}$ and for fixed $f \in X$, the mapping $\phi_f : [0, \infty) \to X$ defined by*

$$\phi_f(t) = T(t)f, \quad t \ge 0$$

is continuous on $[0, \infty)$, i.e., on the right at 0 and both sides for $t > 0$.

Proof. Fix $t \ge 0$ and let $f \in X$. Then, for $\epsilon > 0$, we have $T(t+\epsilon)f = T(\epsilon + t)f = T(\epsilon)[T(t)f] \to T(t)f$ as $\epsilon \to 0+$ by Definition 1.57 (iii) with t and f replaced by ϵ and $T(t)f$ respectively. This shows that ϕ_f is continuous on the right at $t \ge 0$. For left continuity, let $t > 0$ and by the semigroup property we have

$$\begin{aligned}\|T(t)f - T(t-\epsilon)f\| &= \|T(t-\epsilon+\epsilon)f - T(t-\epsilon)f\| \\ &= \|T(t-\epsilon)T(\epsilon)f - T(t-\epsilon)f\| \\ &= \|T(t-\epsilon)[T(\epsilon)f - f]\| \\ &\le \|T(t-\epsilon)\|\|T(\epsilon)f - f\| \to 0, \text{ as } \epsilon \to 0+,\end{aligned}$$

by Definition 1.57 (iii) again, together with Lemma 1.59 we know that $\|T(t-\epsilon)\|$ remains bounded as $\epsilon \to 0+$. □

The conditions (i) and (ii) in Definition 1.57 are reminiscent of the basic properties of the exponential function. More precisely, we have the following result which we can prove easily with some mathematical analysis.

Theorem 1.61 (A functional equation of Cauchy). *Let $f : [0, \infty) \to \mathbb{R}$ be such that*

(i) $f(s+t) = f(s)f(t), \quad \forall s, t \in [0, \infty)$,

(ii) $f(0) = 1$,

(iii) f *is continuous on* $[0, \infty)$ (*on the right at* 0);

then f *has the form*

$$f(t) = e^{tA}, \quad \textit{for some constant } A \in \mathbb{R}.$$

Bearing in mind that we have established Lemma 1.60 as an analogue of condition (iii) in Theorem 1.61, we may now conjecture that the operators $\{T(t)\}_{t \geq 0}$ forming a C_0-semigroup have the form

$$T(t) = e^{tA}, \quad \text{for some operator } A. \tag{1-27}$$

This conjecture leads to many important questions which will form the basis for the rest of this section. Before presenting any rigorous answers, we shall continue our formal analysis in order to provide even stronger motivation for any theory.

From Lemma 1.61 we deduce at once that f is differentiable on $[0, \infty)$ and $\frac{df(t)}{dt} = Ae^{tA} = Af(t)$. Since $f(0) = 1$, we may identify f as the solution of the initial-value problem

$$\frac{du(t)}{dt} = Au, \quad t > 0; \quad u(0) = 1. \tag{1-28}$$

(We could include a one-sided derivative at 0 if we wished.) The analogue of (1-27) might be that the function $\mathcal{U} : [0, \infty) \to X$ defined by $\mathcal{U}(t) = e^{tA}$ satisfies the initial-value problem

$$\frac{d\mathcal{U}(t)}{dt} = A\mathcal{U}(t), \quad t > 0; \quad \mathcal{U}(0) = I. \tag{1-29}$$

The derivative on the left denotes the strong derivative of $\mathcal{U}$ in the $\mathcal{L}(X)$. For fixed $u_0 \in X$, let

$$u(t) = T(t)u_0 = e^{tA}u_0, \quad t \geq 0. \tag{1-30}$$

We might expect this function u to satisfy

$$\frac{du}{dt} = Au, \quad t > 0; \quad u(0) = u_0,$$

where the derivative on the left is the strong derivative of u in X.

Definition 1.62. Let X be a Banach space, $A : D(A) \to X$ be a linear operator and $u : [0, \infty) \to X$. Then we call

$$\begin{cases} \frac{du(t)}{dt} = Au(t), \quad \forall t > 0, \\ u(0) = u_0 \end{cases} \tag{1-31}$$

an abstract Cauchy problem; for short write ACP.

After the analysis above, we now start to make things more rigorous by using the following questions.

(i) Given a C_0-semigroup $\{T(t)\}_{t\geq 0}$, how can we obtain the operator A whose existence is conjectured in (1-27)?
(ii) Which operators A can arise in (1-27)?
(iii) Given a suitable operator A, how can we construct a corresponding semigroup e^{tA}?

To answer these questions, we need to study the infinitesimal generator of a C_0-semigroup and its relation with the C_0-semigroup.

Definition 1.63. Let $\{T(t)\}_{t\geq 0}$ be a C_0-semigroup of linear operators on a Banach space X. The infinitesimal generator of $\{T(t)\}_{t\geq 0}$ is the operator $A : D(A) \to R(A) \subset X$ defined by

$$\begin{aligned} D(A) &= \left\{ f \in X \;\middle|\; \lim_{t\to 0+} A_t f \quad \text{exists in} \quad X \right\}, \\ Af &= \lim_{t\to 0+} A_t f, \quad f \in D(A), \end{aligned} \tag{1-32}$$

where

$$A_t f = \frac{T(t) - I}{t} f = \frac{T(t)f - f}{t}, \quad t > 0, \quad f \in X. \tag{1-33}$$

Remark 1.64. It is easy to check that $D(A)$ is a vector subspace of X and that A is a linear operator.

Theorem 1.65. *Let A be the infinitesimal generator of the C_0-semigroup*

$$\{T(t)\}_{t\geq 0} \subset \mathcal{L}(X).$$

Then, for any $f \in D(A)$,

(i) $T(t)f \in D(A)$ *and* $AT(t)f = T(t)Af, \quad t \in [0, \infty)$,

(ii) $\dfrac{d}{dt}\{T(t)f\} = AT(t)f = T(t)Af.$ $\qquad (1-34)$

The derivative is one-sided at 0 *and two-sided for* $t > 0$.

Proof. (i) Let $f \in D(A)$ and fix $t \geq 0$. Then, for $s > 0$, (1-33) gives

$$\begin{aligned} A_s T(t) f &= \frac{T(s)T(t)f - T(t)f}{s} = \frac{T(s+t)f - T(t)f}{s} \\ &= \frac{T(t)T(s)f - T(t)f}{s} = \frac{T(t)[T(s)f - f]}{s} = T(t)\frac{T(s)f - f}{s}, \end{aligned}$$

where we have used the semigroup property of $T(t)$: $T(t)T(s) = T(s+t)$. As $s \to 0+$, the right-hand side converges to $T(t)(Af)$ since $f \in D(A)$ and $T(t)$ is continuous on X. Hence

$$\lim_{s\to 0+} A_s T(t) f = T(t)Af,$$

by Definition 1.63 which means that

$$T(t)f \in D(A) \quad \text{and} \quad AT(t)f = T(t)Af. \tag{1-35}$$

(ii) For $t \geq 0$ consider the right-hand limit

$$\begin{aligned}
&\lim_{h\to 0+} \frac{T(t+h)f - T(t)f}{h} \\
&= \lim_{h\to 0+} \frac{T(t)T(h)f - T(t)f}{h} \\
&= \lim_{h\to 0+} \frac{T(t)[T(h)f - f]}{h} \\
&= T(t)Af \xlongequal{(1-35)} AT(t)f,
\end{aligned} \tag{1-36}$$

since $T(t)f \in D(A)$ by (i). (1-36) shows that at $t = 0$ (1-34) holds, i.e.,

$$\frac{d}{dt}T(t)f\bigg|_{t=0} = Af. \tag{1-37}$$

For $t > 0$, consider the left-hand limit, by using Lemma 1.59,

$$\begin{aligned}
&\lim_{\delta\to 0+} \left\| \frac{T(t)f - T(t-\delta)f}{\delta} - T(t)Af \right\| \\
&= \lim_{\delta\to 0+} \left\| \frac{T(t-\delta)T(\delta)f - T(t-\delta)f}{\delta} - T(t)Af \right\| \\
&= \lim_{\delta\to 0+} \left\| T(t-\delta)\frac{T(\delta)f - f}{\delta} - T(t)Af \right\| \\
&= \lim_{\delta\to 0+} \left\| T(t-\delta)\frac{T(\delta)f - f}{\delta} - T(t-\delta)Af \right. \\
&\qquad \left. + T(t-\delta)Af - T(t)Af \right\| \\
&\leq \lim_{\delta\to 0+} \|T(t-\delta)\| \left\| \frac{T(\delta)f - f}{\delta} - Af \right\| \\
&\qquad + \lim_{\delta\to 0+} \|T(t-\delta)Af - T(t-\delta)T(\delta)Af\| \\
&\leq \lim_{\delta\to 0+} \mathcal{B}e^{\omega(t-\delta)} \left\| \frac{T(\delta)f - f}{\delta} - Af \right\| \\
&\qquad + \lim_{\delta\to 0+} \mathcal{B}e^{\omega(t-\delta)} \|Af - T(\delta)Af\| \\
&= 0.
\end{aligned} \tag{1-38}$$

From which together with (1-36) and (1-37) it follows that

$$\frac{d}{dt}\{T(t)f\} = T(t)Af = AT(t)f, \quad t \in [0, \infty). \qquad \square$$

Remark 1.66. Let $T(t)$ and $S(t)$ be C_0-semigroups of linear operators with infinitesimal generators A and B, respectively. If $A = B$, then $T(t) = S(t)$. This result follows from Theorem 1.65. In fact, for $\forall f \in D(A) = D(B)$ we have

$$\begin{aligned}
\frac{d}{d\tau}T(t-\tau)S(\tau)f &= -AT(t-\tau)S(\tau)f + T(t-\tau)BS(\tau)f \\
&= -T(t-\tau)AS(\tau)f + T(t-\tau)BS(\tau)f \\
&= 0 \\
&\Longrightarrow \\
0 &= \int_0^t \frac{d}{d\tau}T(t-\tau)S(\tau)f d\tau \\
&\Longrightarrow \\
0 &= T(t-\tau)S(\tau)f\Big|_{\tau=0}^{\tau=t} \\
&= T(0)S(t)f - T(t)S(0)f = S(t)f - T(t)f \\
&\Longrightarrow \\
S(t)f &= T(t)f.
\end{aligned}$$

Theorem 1.67. *Let A be the generator of a C_0-semigroup of linear operators $\{T(t)\}_{t\geq 0}$ in a Banach space X. Then*

(i) *$D(A)$ is a dense vector subspace of X, i.e., $\overline{D(A)} = X$.*
(ii) *$A : D(A) \to R(A) \subset X$ is a closed linear operator.*

Proof. (i) $D(A)$ is a vector subspace of X by Remark 1.64. To prove that $\overline{D(A)} = X$, let $f \in X$ and, for $t > 0$, let

$$g_t = \int_0^t T(s)f ds.$$

By Lemma 1.60, g_t is well defined as an element of X. We shall show that $g_t \in D(A)$. To this end we consider, for $h > 0$,

$$\begin{aligned}
A_h g_t &= \frac{T(h) - I}{h}\int_0^t T(s)f ds \\
&= \frac{1}{h}\int_0^t [T(h) - I]T(s)f ds.
\end{aligned}$$

The last step is permissible because the integral is the limit of a finite sum and the operator $T(h) - I$ is continuous. By the semigroup property

$$\begin{aligned}
A_h g_t &= \frac{1}{h}\int_0^t [T(h+s)f - T(s)f]ds \\
&= \frac{1}{h}\int_0^t T(h+s)f ds - \frac{1}{h}\int_0^t T(s)f ds
\end{aligned}$$

$$\begin{aligned}
&\overset{y=h+s}{=\!=\!=} \frac{1}{h}\int_h^{t+h} T(y)f dy - \frac{1}{h}\int_0^t T(s)f ds \\
&= \frac{1}{h}\left[\int_h^t T(s)f ds + \int_t^{t+h} T(s)f ds\right] \\
&\quad - \frac{1}{h}\left[\int_0^h T(s)f ds + \int_h^t T(s)f ds\right] \\
&= \frac{1}{h}\int_t^{t+h} T(s)f ds - \frac{1}{h}\int_0^h T(s)f ds.
\end{aligned}$$

These manipulations are all valid for numerical-valued integrals. As $h \to 0+$, the right-hand side tends to $T(t)f - T(0)f = T(t)f - f$, by the fundamental theorem of calculus. Hence $g_t \in D(A)$ and $Ag_t = T(t)f - f$.

Now let $f_t = \frac{1}{t} g_t$. Then $f_t \in D(A)$, since $D(A)$ is a vector subspace with $Af_t = \frac{1}{t} Ag_t$. Also

$$f_t = \frac{1}{t}\int_0^t T(s)f ds \to T(0)f = f, \quad \text{as} \quad t \to 0+ .$$

Since f is an arbitrary element of X, $\overline{D(A)} = X$.
(ii) To prove that A is closed, let $\{f_n \mid n \geq 1\} \subset D(A)$ with $f_n \to f, \quad Af_n \to g$ as $n \to \infty$. By (1-34) we obtain, for fixed $t > 0$,

$$\int_0^t \frac{d}{ds} T(s) f_n ds = \int_0^t T(s) A f_n ds$$

$$\Longrightarrow$$

$$T(s) f_n \big|_0^t = \int_0^t T(s) A f_n ds$$

$$\Longrightarrow$$

$$T(t) f_n - f_n = \int_0^t T(s) A f_n ds, \quad n \geq 1,\ t \in [0, \infty). \tag{1-39}$$

Since by Lemma 1.60 the mapping $s \to T(s)f$ is continuous on $[0, t]$, we have $\|T(s)\| \leq \mathcal{B}$ for all $s \in [0, t]$, where $\mathcal{B}$ is a constant (see Lemma 1.59). Hence

$$\begin{aligned}
&\left\| \int_0^t T(s) A f_n ds - \int_0^t T(s) g ds \right\| \\
&\leq \int_0^t \|T(s)\| \|Af_n - g\| ds \\
&\leq \mathcal{B} \int_0^t \|Af_n - g\| ds \\
&\leq \mathcal{B} t \|Af_n - g\| \to 0, \quad \text{as} \quad n \to \infty.
\end{aligned}$$

Letting $n \to \infty$ in (1-39) therefore leads to

$$T(t)f - f = \int_0^t T(s)gds$$
$$\Longrightarrow$$
$$\frac{1}{t}\{T(t)f - f\} = \frac{1}{t}\int_0^t T(s)gds.$$

As above the right-hand side tends to $T(0)g = g$ as $t \to 0+$, since the left-hand side is $A_t f$, we deduce, on letting $t \to 0+$, that $f \in D(A)$ and $Af = g$. This implies that A is closed. □

Theorem 1.68 (Hille-Yosida). *A linear operator A is the infinitesimal generator of a C_0-semigroup of contractions $T(t)$, $t \in [0,\infty)$ if and only if*

(i) *A is closed and $\overline{D(A)} = X$,*

(ii) *The resolvent set $\rho(A)$ of A contains $(0,\infty)$ and, for $\gamma > 0$,*

$$\|R(\gamma, A)\| = \left\|(\gamma I - A)^{-1}\right\| \le \frac{1}{\gamma}. \tag{1-40}$$

Proof. Necessity. If A is the infinitesimal generator of a C_0-semigroup, then it is closed and $\overline{D(A)} = X$ by Theorem 1.67. For $\gamma > 0$ and $f \in X$, let

$$R(\gamma)f = \int_0^\infty e^{-\gamma t}T(t)fdt. \tag{1-41}$$

Since $f \to T(t)f$ is continuous by Lemma 1.60 and uniformly bounded $\|T\| \le 1$, the integral exists as an improper Riemann integral and defines a bounded linear operator $R(\gamma)$ satisfying

$$\begin{aligned}\|R(\gamma)f\| &\le \int_0^\infty e^{-\gamma t}\|T(t)f\|dt \\ &\le \int_0^\infty e^{-\gamma t}\|T(t)\|\|f\|dt \\ &\le \|f\|\int_0^\infty e^{-\gamma t}dt \\ &= \frac{1}{\gamma}\|f\|. \end{aligned} \tag{1-42}$$

Furthermore, by the semigroup property we have, for $h > 0$,

$$\begin{aligned}\frac{T(h) - I}{h}R(\gamma)f &= \frac{T(h) - I}{h}\int_0^\infty e^{-\gamma t}T(t)fdt \\ &= \frac{1}{h}\int_0^\infty e^{-\gamma t}[T(h) - I]T(t)fdt\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{h}\int_0^\infty e^{-\gamma t}[T(t+h)f - T(t)f]dt \\
&= \frac{1}{h}\int_0^\infty e^{-\gamma t}T(t+h)f dt \\
&\quad - \frac{1}{h}\int_0^\infty e^{-\gamma t}T(t)f dt \\
&\xlongequal{y=t+h} \frac{1}{h}\int_h^\infty e^{-\gamma(y-h)}T(y)f dy \\
&\quad - \frac{1}{h}\int_0^\infty e^{-\gamma t}T(t)f dt \\
&= \frac{1}{h}\int_0^\infty e^{-\gamma(y-h)}T(y)f dy \\
&\quad - \frac{1}{h}\int_0^h e^{-\gamma(y-h)}T(y)f dy \\
&\quad - \frac{1}{h}\int_0^\infty e^{-\gamma t}T(t)f dt \\
&= \frac{e^{\gamma h}}{h}\int_0^\infty e^{-\gamma t}T(t)f dt \\
&\quad - \frac{1}{h}\int_0^\infty e^{-\gamma t}T(t)f dt \\
&\quad - \frac{1}{h}\int_0^h e^{-\gamma(t-h)}T(t)f dt \\
&= \frac{e^{\gamma h}-1}{h}\int_0^\infty e^{-\gamma t}T(t)f dt \\
&\quad - \frac{1}{h}\int_0^h e^{-\gamma(t-h)}T(t)f dt \\
&= \frac{e^{\gamma h}-1}{h}\int_0^\infty e^{-\gamma t}T(t)f dt \\
&\quad - e^{\gamma h}\frac{1}{h}\int_0^h e^{-\gamma t}T(t)f dt. \qquad (1\text{-}43)
\end{aligned}
$$

As $h \to 0+$, the right-hand side of (1-43) converges to $\gamma R(\gamma)f - f$. This implies that for every $f \in X$ and $\gamma > 0$, $R(\gamma)f \in D(A)$ and $AR(\gamma) = \gamma R(\gamma) - I$ or

$$(\gamma I - A)R(\gamma) = I. \qquad (1\text{-}44)$$

For $f \in D(A)$ and by Theorem 1.65 we have

$$R(\gamma)Af = \int_0^\infty e^{-\gamma t}T(t)Af dt = \int_0^\infty e^{-\gamma t}AT(t)f dt$$

$$= A\left(\int_0^\infty e^{-\gamma t}T(t)f\,dt\right) = AR(\gamma)f. \tag{1-45}$$

Here we used the closedness of A. From (1-44) and (1-45) it follows that

$$(\gamma I - A)R(\gamma) = R(\gamma)(\gamma I - A)f = f, \quad f \in D(A). \tag{1-46}$$

Thus, $R(\gamma)$ is the inverse of $\gamma I - A$, that is, $R(\gamma) = R(\gamma, A)$, it exists for all $\gamma > 0$ and satisfies the desired estimate (1-40) (see (1-42)). Conditions (i) and (ii) are therefore necessary.

Sufficiency. We split the proof into five steps. Firstly, we show that

$$\lim_{\gamma\to\infty} \gamma R(\gamma, A)f = f, \quad f \in X. \tag{1-47}$$

In fact, if $f \in D(A)$, then by using

$$\begin{aligned}&\gamma R(\gamma, A) - I = AR(\gamma, A),\\ &A(\gamma I - A) = (\gamma I - A)A \Rightarrow R(\gamma, A)A = AR(\gamma, A)\end{aligned}$$

and (1-40) we have

$$\begin{aligned}\|\gamma R(\gamma, A)f - f\| &= \|AR(\gamma, A)f\| = \|R(\gamma, A)Af\|\\ &\le \frac{1}{\gamma}\|Af\|, \quad \text{as} \quad \gamma > 0.\end{aligned} \tag{1-48}$$

Because $D(A)$ is dense in X and $\|\gamma R(\gamma, A)\| \le 1$, $\gamma R(\gamma, A)f \to f$ as $\gamma \to \infty$ for every $f \in X$.

Secondly, we verify that the Yosida approximation

$$\begin{aligned}A_\gamma &= \gamma AR(\gamma, A) = \gamma[\gamma R(\gamma, A) - I]\\ &= \gamma^2 R(\gamma, A) - \gamma I\end{aligned}$$

satisfies

$$\lim_{\gamma\to\infty} A_\gamma f = Af, \quad f \in D(A). \tag{1-49}$$

In fact, for $f \in D(A)$, by (1-47), $R(\gamma, A)A = AR(\gamma, A)$ and the definition of A_γ it follows that

$$\lim_{\gamma\to\infty} A_\gamma f = \lim_{\gamma\to\infty} \gamma R(\gamma, A)Af = Af.$$

Thirdly, we will check that A_γ is the infinitesimal generator of a uniformly continuous semigroup of contractions e^{tA_γ} and furthermore, for every $f \in X$, γ, $\beta > 0$,

$$\left\|e^{tA_\gamma}f - e^{tA_\beta}f\right\| \le t\|A_\gamma f - A_\beta f\|. \tag{1-50}$$

From (1-48) and the definition of A_γ it is evident that A_γ is a bounded linear operator and thus is the infinitesimal generator of a uniformly continuous semigroup

e^{tA_γ} of bounded linear operators (see Remark 1.58 (iv)). Also, by (1-40) and the Taylor expansion of an exponential function (see Remark 1.58 (iv)),

$$\begin{aligned}\|e^{tA_\gamma}\| &= \left\|e^{t(\gamma^2 R(\gamma,A)-\gamma I)}\right\| = \left\|e^{t\gamma^2 R(\gamma,A)}e^{-\gamma t}\right\| \\ &= e^{-\gamma t}\left\|e^{t\gamma^2 R(\gamma,A)}\right\| \le e^{-\gamma t}e^{\gamma^2 t\|R(\gamma,A)\|} \\ &\le e^{-\gamma t}e^{\gamma^2 t\frac{1}{\gamma}} = e^{-\gamma t}e^{\gamma t} \\ &= 1 \end{aligned} \tag{1-51}$$

and therefore e^{tA_γ} is a semigroup of contractions. It is obvious that e^{tA_γ}, e^{tA_β}, A_γ and A_β commute with each other. Consequently, by (1-51),

$$\begin{aligned}&\left\|e^{tA_\gamma}f - e^{tA_\beta}f\right\| \\ &= \left\|\int_0^1 \frac{d}{ds}\left(e^{tsA_\gamma}e^{t(1-s)A_\beta}f\right)ds\right\| \\ &\le \int_0^1 \left\|\frac{d}{ds}\left(e^{tsA_\gamma}e^{t(1-s)A_\beta}f\right)\right\| ds \\ &= \int_0^1 \left\|tA_\gamma e^{tsA_\gamma}e^{t(1-s)A_\beta}f - tA_\beta e^{tsA_\gamma}e^{t(1-s)A_\beta}f\right\| ds \\ &= \int_0^1 t\left\|e^{tsA_\gamma}e^{t(1-s)A_\beta}(A_\gamma f - A_\beta f)\right\| ds \\ &\le t\int_0^1 \left\|e^{tsA_\gamma}e^{t(1-s)A_\beta}\right\| \|A_\gamma f - A_\beta f\| ds \\ &\le t\|A_\gamma f - A_\beta f\| \int_0^1 \|e^{tsA_\gamma}\| \left\|e^{t(1-s)A_\beta}\right\| ds \\ &\le t\|A_\gamma f - A_\beta f\| \int_0^1 ds \\ &= t\|A_\gamma f - A_\beta f\|.\end{aligned}$$

Fourthly, by using A_γ we will construct a C_0-semigroup. Let $f \in D(A)$. Then by (1-49) and (1-50),

$$\begin{aligned}\|e^{tA_\gamma}f - e^{tA_\beta}f\| &\le t\|A_\gamma f - A_\beta f\| \\ &\le t\|A_\gamma f - Af\| + t\|A_\beta f - Af\|.\end{aligned} \tag{1-52}$$

Which shows that $e^{tA_\gamma}f$ is a Cauchy sequence in the Banach space X. From (1-52) and (1-49) it follows that for $f \in D(A)$, $e^{tA_\gamma}f$ converges as $\gamma \to \infty$ and the convergence is uniform in bounded intervals. Since $D(A)$ is dense in X and $\|e^{tA_\gamma}\| \le 1$, it follows that

$$\lim_{\gamma\to\infty} e^{tA_\gamma}f \xlongequal{\text{set}} T(t)f, \quad f \in X. \tag{1-53}$$

The limit in (1-53) is again uniform on bounded intervals. From (1-53) it follows readily that the limit $T(t)$ satisfies the semigroup property:

$$T(0) = I, \quad T(s)T(t) = T(s+t) = T(t)T(s).$$

Also $t \to T(t)f$ is continuous for $t \geq 0$ as a uniform limit of the continuous functions $t \to e^{tA_\gamma} f$. (1-51) shows that $\|T(t)\| \leq 1$. Hence, $T(t)$ is a C_0-semigroup of contractions on X.

Lastly, we will show the uniqueness of the C_0-semigroup. Let $f \in D(A)$. Then by using (1-49), (1-53) and Theorem 1.65 we have

$$\begin{aligned} T(t)f - f &= \lim_{\gamma\to\infty} \left(e^{tA_\gamma} f - f\right) = \lim_{\gamma\to\infty} e^{sA_\gamma} f\big|_{s=0}^{s=t} \\ &= \lim_{\gamma\to\infty} \int_0^t \frac{d}{ds} e^{sA_\gamma} f ds = \lim_{\gamma\to\infty} \int_0^t e^{sA_\gamma} A_\gamma f ds \\ &= \int_0^t T(s)Af ds. \end{aligned} \tag{1-54}$$

The last equality follows from the uniform convergence of $e^{tA_\gamma} A_\gamma f$ to $T(t)Af$ on bounded intervals. Let B be the infinitesimal generator of $T(t)$ and $f \in D(A)$. Dividing (1-54) by $t > 0$ and $t \to 0$ we see that $f \in D(B)$ and $Bf = Af$. Thus $D(A) \subset D(B)$ and $B|_{D(A)} = A$ (see Definition 1.4). Since B is the infinitesimal generator of $T(t)$, it follows from the necessary conditions that $1 \in \rho(B)$. On the other hand, we assume that $1 \in \rho(A)$. Since $D(A) \subset D(B)$ and $B|_{D(A)} = A$, $(I-B)D(A) = (I-A)D(A) = X$ which implies $D(B) = (I-B)^{-1}X = D(A)$ and therefore $A = B$. $\square$

From the Hille-Yosida theorem (see Theorem 1.68) it is easy to see some simple consequences.

Corollary 1.69. *Let A be the infinitesimal generator of a C_0-semigroup of contractions $T(t)$. Then the resolvent set of A contains the open right half complex plane, i.e., $\{\gamma \in \mathbb{C} \mid \Re\gamma > 0\} \subset \rho(A)$ and for such γ,*

$$\|R(\gamma, A)\| \leq \frac{1}{\Re\gamma}. \tag{1-55}$$

Let $T(t)$ be a C_0-semigroup satisfying $\|T(t)\| \leq e^{\omega t}$ (for some $\omega \geq 0$). Consider $S(t) = e^{-\omega t}T(t)$. $S(t)$ is obviously a C_0-semigroup of contractions. If A is the infinitesimal generator of $T(t)$, then $A - \omega I$ is the infinitesimal generator of $S(t)$. On the other hand, if A is the infinitesimal generator of a C_0-semigroup of contractions $S(t)$, then $A + \omega I$ is the infinitesimal generator of a C_0-semigroup $T(t)$ satisfying $\|T(t)\| \leq e^{\omega t}$. Indeed, $T(t) = e^{\omega t}S(t)$. These remarks lead us to the characterization of the infinitesimal generators of a C_0-semigroup satisfying $\|T(t)\| \leq e^{\omega t}$.

Corollary 1.70. *A linear operator A is the infinitesimal generator of a C_0-semigroup satisfying $\|T(t)\| \le e^{\omega t}$ if and only if*

(i) A *is closed and* $\overline{D(A)} = X$,

(ii) *The resolvent set $\rho(A)$ of A contains the ray $\{\gamma \in \mathbb{C} \mid \Im\gamma = 0, \quad \Re\gamma > \omega\}$ and for such γ,*

$$\|R(\gamma, A)\| \le \frac{1}{\gamma - \omega}.$$

Similar to Corollary 1.70 we can describe the C_0-semigroup $\|T(t)\| \le \mathcal{B}e^{\omega t}$. Other parts are the same as Corollary 1.70 except for the last inequality, which we modify as follows (see Engel and Nagel [27]).

$$\|[R(\gamma, A)]^n\| \le \frac{\mathcal{B}}{(\gamma - \omega)^n}. \tag{1-56}$$

Let X^* be the dual space of the Banach space X. We denote the value of $f^* \in X^*$ at $f \in X$ by $\langle f^*, f\rangle$ or $\langle f, f^*\rangle$. X^* is a Banach space under the norm

$$|||f^*||| = \sup_{\substack{\|f\| \le 1 \\ f \in X}} |\langle f^*, f\rangle|.$$

(see Proposition 1.19). For every $f \in X$ we define the duality set $F(f) \subset X^*$ by

$$F(f) = \left\{f^* \in X^* \mid \langle f^*, f\rangle = \|f\|^2 = |||f^*|||^2\right\}. \tag{1-57}$$

From the Hahn-Banach extension theorem (see Theorem 1.16, Corollary 1.17) it follows that $F(f) \neq \emptyset$ for every $f \in X$.

Definition 1.71. A linear operator A is dissipative if for every $f \in D(A)$ there is an $f^* \in F(f)$ such that $\Re\langle Af, f^*\rangle \le 0$.

A useful characterization of dissipative operators is given next.

Theorem 1.72. *A linear operator A is dissipative if and only if*

$$\|(\gamma I - A)f\| \ge \gamma\|f\|, \quad f \in D(A), \quad \gamma > 0. \tag{1-58}$$

Proof. Necessity. Let A be dissipative, $\gamma > 0$ and $f \in D(A)$. If $f^* \in F(f)$ (see (1-57)) and $\Re\langle Af, f^*\rangle \le 0$ by Definition 1.71, then

$$\begin{aligned}
\|\gamma f - Af\|\|f\| &\ge |\langle \gamma f - Af, f^*\rangle| \\
&\ge \Re\langle(\gamma I - A)f, f^*\rangle \\
&= \Re\left\{\langle\gamma f, f^*\rangle - \langle Af, f^*\rangle\right\} \\
&= \gamma\|f\|^2 - \Re\langle Af, f^*\rangle \ge \gamma\|f\|^2
\end{aligned}$$

and (1-58) follows at once.

Sufficiency. Let $f \in D(A)$ and assume that $\gamma\|f\| \le \|\gamma f - Af\|$ for all $\gamma > 0$. If $g_\gamma^* \in F(\gamma f - Af)$ and $q_\gamma^* = \frac{g_\gamma^*}{|||g_\gamma^*|||}$, then $|||q_\gamma^*||| = 1$ and

$$\begin{aligned}\gamma\|f\| &\le \|\gamma f - Af\| = \langle \gamma f - Af, q_\gamma^*\rangle \\ &= \gamma\Re\langle f, q_\gamma^*\rangle - \Re\langle Af, q_\gamma^*\rangle \\ &\le \gamma\|f\| - \Re\langle Af, q_\gamma^*\rangle\end{aligned}$$

for every $\gamma > 0$. Therefore, by

$$\Re\langle Af, q_\gamma^*\rangle \le |\langle Af, q_\gamma^*\rangle| \le \|Af\| |||q_\gamma^*||| = \|Af\|$$

we have

$$\begin{aligned}&\Re\langle Af, q_\gamma^*\rangle \le 0 \quad \text{and} \\ &\Re\langle f, q_\gamma^*\rangle \ge \|f\| - \frac{1}{\gamma}\Re\langle Af, q_\gamma^*\rangle \ge \|f\| - \frac{1}{\gamma}\|Af\|.\end{aligned} \tag{1-59}$$

Since the unit ball of X^* is compact in the weak-star topology of X^* by Theorem 1.39, the net q_γ^* has a weak-star cluster point $q^* \in X^*$, $|||q^*||| \le 1$. From (1-59) it follows that $\Re\langle Af, q^*\rangle \le 0$ and $\Re\langle f, q^*\rangle \ge \|f\|$. But $\Re\langle f, q^*\rangle \le |\langle f, q^*\rangle| \le \|f\|$ and therefore $\langle f, q^*\rangle = \|f\|$. Taking $f^* = \|f\|q^*$ we have $f^* \in F(f)$ and $\Re\langle Af, f^*\rangle \le 0$. Thus for every $f \in D(A)$ there is an $f^* \in F(f)$ such that $\Re\langle Af, f^*\rangle \le 0$ and A is dissipative. □

Theorem 1.73 (Lumer-Phillips). *Let A be a linear operator with dense domain $D(A)$ in X.*

(i) *If A is dissipative and there is a $\gamma_0 > 0$ such that $R(\gamma_0 I - A)$, the range of $\gamma_0 I - A$, is X, then A is the infinitesimal generator of a C_0-semigroup of contractions on X.*

(ii) *If A is the infinitesimal generator of a C_0-semigroup of contractions on X, then $R(\gamma I - A) = X$ for all $\gamma > 0$ and A is dissipative. Moreover, for every $f \in D(A)$ and every $f^* \in F(f)$, $\Re\langle Af, f^*\rangle \le 0$.*

Proof. Let $\gamma > 0$; the dissipativeness of A implies by Theorem 1.72 that

$$\|\gamma f - Af\| \ge \gamma\|f\|, \quad \gamma > 0, f \in D(A). \tag{1-60}$$

Since $R(\gamma_0 I - A) = X$, it follows from (1-60) with $\gamma = \gamma_0$ that $(\gamma_0 I - A)^{-1}$ is a bounded linear operator and thus closed. In fact, we take $f = (\gamma_0 I - A)^{-1}g$ in (1-60), then $\|(\gamma_0 I - A)^{-1}\| \le \frac{1}{\gamma_0}$. But then $\gamma_0 I - A$ is closed and therefore also A is closed. If $R(\gamma I - A) = X$ for every $\gamma > 0$, then $(0, \infty) \subset \rho(A)$ and $\|R(\gamma, A)\| \le \frac{1}{\gamma}$. It then follows from the Hille-Yosida theorem (see Theorem 1.68) that A is the infinitesimal generator of a C_0-semigroup of contractions on X.

To complete the proof of (i) it remains to show that $R(\gamma I - A) = X$ for all $\gamma > 0$, consider the set

$$\Omega = \{\gamma \in (0, \infty) \mid R(\gamma I - A) = X\}.$$

Let $\gamma \in \Omega$. By (1-60), $\gamma \in \rho(A)$. Since $\rho(A)$ is open by Theorem 1.25, a neighborhood of γ is in $\rho(A)$. The intersection of this neighborhood with a real line is clearly in Ω and therefore Ω is open. On the other hand, let $\gamma_n \in \Omega$, $\gamma_n \to \gamma > 0$. For every $g \in X$ there exists an $f_n \in D(A)$ such that

$$\gamma_n f_n - Af_n = g. \tag{1-61}$$

From (1-60) it follows that $\|f_n\| \le \frac{1}{\gamma_n}\|g\| \le \mathcal{D}$ for some $\mathcal{D} > 0$. Now, by (1-60) and (1-61) we have

$$\begin{aligned}
\gamma_m\|f_n - f_m\| &\le \|\gamma_m(f_n - f_m) - A(f_n - f_m)\| \\
&= \|\gamma_m f_n - Af_n - \gamma_m f_m + Af_m\| \\
&= \|\gamma_m f_n + (g - \gamma_n f_n) - g\| \\
&= \|\gamma_m f_n - \gamma_n f_n\| \\
&= \|(\gamma_m - \gamma_n)f_n\| \\
&\le |\gamma_m - \gamma_n|\|f_n\| \\
&\le \mathcal{D}|\gamma_m - \gamma_n|.
\end{aligned} \tag{1-62}$$

Therefore $\{f_n\}$ is a Cauchy sequence (see Definition 1.10). Because X is a Banach space (see Definition 1.10), $f_n \to f \in X$. Then by (1-61) $Af_n \to \gamma f - g$. Since A is closed, $f \in D(A)$ and $\gamma f - Af = g$. Hence, $R(\gamma I - A) = X$ and $\gamma \in \Omega$. Thus Ω is also closed in $(0, \infty)$ and since $\gamma_0 \in \Omega$ by assumption $\Omega \ne \emptyset$ and therefore $\Omega = (0, \infty)$. This completes the proof of (i).

If A is the infinitesimal generator of a C_0-semigroup of contractions $T(t)$ on X, then by the Hille-Yosida theorem (see Theorem 1.68) $(0, \infty) \subset \rho(A)$ and $R(\gamma I - A) = X$ for all $\gamma > 0$. Furthermore, if $f \in D(A)$ and $f^* \in F(f)$, then

$$|\langle T(t)f, f^*\rangle| \le \|T(t)f\|\|f^*\| \le \|f\|\|f^*\| = \|f\|^2$$

and therefore

$$\Re\langle T(t)f - f, f^*\rangle = \Re\langle T(t)f, f^*\rangle - \|f\|^2 \le 0. \tag{1-63}$$

Dividing (1-63) by $t > 0$ and by letting $t \to 0$ yields

$$\Re\langle Af, f^*\rangle \le 0, \tag{1-64}$$

which holds for every $f^* \in F(f)$. $\square$

Definition 1.74. Let X be a Banach lattice (see Definition 1.10). An operator A is called dispersive if

$$\langle Af, f^+\rangle \le 0, \quad f \in D(A);$$

here $f^+ = \sup\{f, 0\}$, $f^- = \sup\{-f, 0\}$, $f = f^+ - f^-$, $|f| = \sup\{f, -f\} = f^+ + f^-$.

Lemma 1.75. *Let* X *be a Banach lattice* (*see Definition* 1.10). *Given* $f \geq 0$, *there exists an* $f^* \in X^*$ *satisfying*

(i) f^* *is positive.*
(ii) $\langle f^*, f \rangle = \|f\|^2 = \|f^*\|^2$.
(iii) $\langle f^*, g \rangle = 0$ *for every* g *such that* $\min\{f, |g|\} = 0$.

Proof. Setting $\overline{Y} = \{g \mid \min\{f, |g|\} = 0\}$ it can be shown that $\overline{Y}$ is a closed linear subspace and that if $|q| \leq |g|$ for $g \in \overline{Y}$, then $q \in \overline{Y}$. Moreover $\|f - g\| \geq \|f\|$ for all $g \in \overline{Y}$. In fact, according to the definition of $|\cdot|$,

$$\begin{aligned}\max\{f, g\} - \min\{f, g\} &= \frac{1}{2}\{f + g + |f - g|\} - \frac{1}{2}\{f + g - |f - g|\} \\ &= |f - g|\end{aligned}$$

and since $\max\{f, g\} \geq f$ and $\min\{f, g\} \leq \min\{f, |g|\} = 0$, we see that $|f - g| \geq f$ and hence the assertion holds. By the Hahn-Banach extension theorem (see Theorem 1.16) and Corollary 1.17 there exists an $f^* \in X^*$ such that $\|f^*\| = \|f\|$, $\langle f^*, f \rangle = \|f\|^2$, and $\langle f^*, \overline{Y} \rangle = 0$. Next we decompose f^* into its positive and negative parts: $f^* = f_*^+ - f_*^-$ where for $f \geq 0$, $\langle f_*^+, f \rangle = \sup\{\langle f^*, g \rangle \mid 0 \leq g \leq f\}$. It is clear from the above-stated properties of $\overline{Y}$ that $\langle f_*^+, \overline{Y} \rangle = 0$. Furthermore, for arbitrary $g \in X$, we have

$$\begin{aligned}|\langle f_*^+, g \rangle| &= |\langle f_*^+, g^+ \rangle - \langle f_*^+, g^- \rangle| \\ &\leq \max\{|\langle f_*^+, g^+ \rangle|, |\langle f_*^+, g^- \rangle|\} \\ &\leq \|f^*\| \max\{\|g^+\|, \|g^-\|\} \\ &\leq \|f^*\| \|g\|,\end{aligned}$$

here $g^- = \sup\{-g, 0\} = -\inf\{g, 0\}$. So that $\|f_*^+\| \leq \|f^*\|$. Finally, for the given f,

$$\begin{aligned}\langle f^*, f \rangle &\leq \langle f_*^+, f \rangle \leq \|f_*^+\| \|f\| \\ &\leq \|f^*\| \|f\| = \|f\|^2 = \langle f^*, f \rangle\end{aligned}$$

and consequently $\langle f_*^+, f \rangle = \langle f^*, f \rangle = \|f\|^2$ and $\|f_*^+\| = \|f\|$. It follows that f_*^+ satisfies the assertion of this lemma. □

Lemma 1.76. *Let* X *be a Banach lattice* (*see Definition* 1.10). *If* T *is a linear positive operator contractive on positive elements, that is* $\|Tf\| \leq \|f\|$ *for* $f \geq 0$, *then* T *is a contraction operator, i.e.,* $\|T(t)f\| \leq \|f\|, \quad \forall f \in X$.

Proof. Since $|g + q| \leq |g| + |q|$, we see that

$$\begin{aligned}|Tf| &= |Tf^+ - Tf^-| \leq |Tf^+| + |Tf^-| \\ &= Tf^+ + Tf^- = T(f^+ + f^-) = T|f|.\end{aligned}$$

Hence, by the definition of Banach lattice (see Definition 1.10):

$$|f| \le |g| \Longrightarrow \|f\| \le \|g\|, \quad f,\ g \in X$$

we deduce the desired result:

$$\|Tf\| \le \|T|f|\| \le \||f|\| = \|f\|. \qquad \square$$

Theorem 1.77 (Phillips). *Let X be a Banach lattice. A necessary and sufficient condition for a linear operator A with dense domain to generate a C_0-semigroup of positive contraction operators is that A be dispersive with $R(I - A) = X$.*

Proof. If A generates a C_0-semigroup of positive contraction operators $T(t)$, then $R(I - A) = X$ by the Hille-Yosida theorem (see Theorem 1.68), and further by Lemma 1.75,

$$\begin{aligned}
\langle f, f^+\rangle &= \|f^+\|^2 = \|f^+\|\|f^+\| \\
&\ge \|T(t)f^+\|\|f^+\| \ge \langle T(t)f^+, f^+\rangle \\
&\ge \langle T(t)f^+, f^+\rangle - \langle T(t)f^-, f^+\rangle \\
&= \langle T(t)(f^+ - f^-), f^+\rangle \\
&= \langle T(t)f, f^+\rangle,
\end{aligned}$$

so that

$$\begin{aligned}
\langle Af, f^+\rangle &= \left\langle \lim_{t\to 0+} \frac{T(t)f - f}{t}, f^+ \right\rangle \\
&= \lim_{t\to 0+} \left\{ \frac{1}{t}\langle T(t)f, f^+\rangle - \frac{1}{t}\langle f, f^+\rangle \right\} \\
&= \lim_{t\to 0+} \frac{1}{t}\{\langle T(t)f, f^+\rangle - \langle f, f^+\rangle\} \\
&\le 0
\end{aligned}$$

which proves that A is dispersive.

In order to prove the converse assertion, let us suppose for the moment that $R(\gamma I - A) = X$ for some $\gamma > 0$. Then for fixed $g > 0$ in X there is an $f \in D(A)$ such that $\gamma f - Af = g$. Making use of the dispersive property of A and by Lemma 1.75 we see that

$$\begin{aligned}
\gamma\|f^-\|^2 &= \gamma\|\sup\{-f, 0\}\|^2 = \gamma\|(-f)^+\|^2 = \gamma\langle -f, (-f)^+\rangle \\
&\le \gamma\langle -f, (-f)^+\rangle - \langle A(-f), (-f)^+\rangle \\
&= \langle -\gamma f - A(-f), (-f)^+\rangle = \langle -(\gamma f - Af), (-f)^+\rangle \\
&= \langle -g, (-f)^+\rangle \le 0,
\end{aligned}$$

i.e., $f^- = 0$, consequently $f \geq 0$ and

$$\begin{aligned}\gamma\|f\|^2 &= \gamma\langle f, f^+\rangle \leq \gamma\langle f, f^+\rangle - \langle Af, f^+\rangle \\ &= \langle \gamma f - Af, f^+\rangle = \langle g, f^+\rangle \leq \|g\|\|f^+\| \\ &= \|g\|\|f\|.\end{aligned}$$

Thus

$$\gamma\|f\| \leq \|g\| \Longrightarrow \|f\| \leq \frac{1}{\gamma}\|g\|. \tag{1-65}$$

Since 0 is a nonnegative element, the relation (1-65) implies that $\gamma I - A$ is one-to-one. Hence, (1-65) implies that $\gamma R(\gamma, A)$ is positive and

$$\|\gamma R(\gamma, A)\| = \|\gamma(\gamma I - A)^{-1}\| = \gamma\|(\gamma I - A)^{-1}\| \leq \gamma\frac{1}{\gamma} = 1.$$

Now according to

$$R(\beta, A) = R(\gamma, A)[I + (\beta - \gamma)R(\gamma, A)]^{-1}$$

holds for $|\beta - \gamma| < \gamma$. In particular then, $R(\beta I - A) = X$ for $|\beta - \gamma| < \gamma$ and the dispersive property shows as above that $\beta R(\beta, A)$ is a positive contraction operator in this range. This permits us to extend the result by analytic continuation to all $\beta > 0$; it is known that $R(\gamma I - A) = X$ for some $\gamma > 0$. However this is precisely what is assumed in the hypothesis to this theorem. The Hille-Yosida theorem (see Theorem 1.68) therefore applies and establishes the fact that A is the generator of a strongly continuous semigroup of contraction operators $T(t)$. It is evident from the proof of the Hille-Yosida theorem (see (1-53)) that

$$\begin{aligned}T(t)f &= \lim_{\gamma\to\infty} e^{tA_\gamma} f = \lim_{\gamma\to\infty} e^{t[\gamma^2 R(\gamma,A)-\gamma I]} f \\ &= \lim_{\gamma\to\infty} e^{-\gamma t} e^{\gamma^2 R(\gamma,A)t} f = \lim_{\gamma\to\infty} e^{-\gamma t} \sum_{n=0}^{\infty} \frac{[\gamma^2 R(\gamma, A)t]^n}{n!} f \\ &= \lim_{\gamma\to\infty} e^{-\gamma t} \sum_{n=0}^{\infty} \frac{(\gamma t)^n [\gamma R(\gamma, A)]^n f}{n!}\end{aligned}$$

and it follows from this expression that $T(t)$ is a positive operator if $\gamma R(\gamma, A)$ is positive. □

By combining the Phillips theorem with the Lumer-Phillips theorem we obtain a consequence as follows.

Corollary 1.78. *If X is a Banach lattice and A is a dispersive operator which generates a positive contraction C_0-semigroup, then A is also dissipative.*

Although the Phillips theorem and the Lumer-Phillips theorem are equivalent when X is a Banach lattice, it is convenient to prove the conditions of the Phillips theorem. In the case when $X = L^1[0,\infty)$, Example 1.1(c) in Nagel [91] (page 248) shows us how to get a positive contraction C_0-semigroup $T(t)$ from a given operator A. It states that, if $f \in X$, then

$$\phi \in \{g \in (L^1[0,\infty))^+ \mid \|g\| \le 1,\ \langle f, g\rangle = \|f^+\|\}$$

if and only if

$$\phi(x) = \begin{cases} 1 & \text{if} \quad f(x) > 0, \\ 0 \le \phi(x) \le 1 & \text{if} \quad f(x) = 0, \\ 0 & \text{if} \quad f(x) < 0. \end{cases} \tag{1-66}$$

Through combining the Phillips theorem with (1-66), we obtain a powerful and convenient theorem to study reliability models in the Banach space $L^1[0,\infty)$.

Theorem 1.79. *Let A be a densely defined linear operator on $L^1[0,\infty)$. The following assertions are equivalent.*

(i) *A is the generator of a positive contraction C_0-semigroup.*

(ii)
$$(\gamma I - A)f = g \in L^1[0,\infty) \tag{1-67}$$

has a solution $f \in D(A)$, and for $p \in D(A)$

$$\langle Ap, \phi\rangle \le 0, \tag{1-68}$$

where

$$\phi(x) = \begin{cases} 1 & \textit{if} \quad p(x) > 0, \\ 0 \le \phi(x) \le 1 & \textit{if} \quad p(x) = 0, \\ 0 & \textit{if} \quad p(x) < 0. \end{cases} \tag{1-69}$$

Theorem 1.80 (Perturbation of a C_0-semigroup). *Let X be a Banach space and A be the infinitesimal generator of a C_0-semigroup $T(t)$ on X, satisfying $\|T(t)\| \le \mathcal{B}e^{\omega t}$. If $\mathcal{G}$ is a bounded linear operator on X, then $A + \mathcal{G}$ is the infinitesimal generator of a C_0-semigroup $S(t)$ on X satisfying $\|S(t)\| \le \mathcal{B}e^{(\omega+\mathcal{B}\|\mathcal{G}\|)t}$.*

Proof. If we define the new norm of f as

$$|f| = \lim_{\gamma\to\infty} \sup_{n\ge 0} \|\gamma^n [R(\gamma, A)]^n f\|,$$

then it is not difficult to prove by (1-56) that $|\cdot|$ and $\|\cdot\|$ are equivalent norms on X. In fact,

$$\|f\| = \|\gamma^0 [R(\gamma, A)]^0 f\| \le \lim_{\gamma\to\infty} \sup_{n\ge 0} \|\gamma^n [R(\gamma, A)]^n f\|$$

$$
\begin{aligned}
&= |f| \le \lim_{\gamma\to\infty} \sup_{n\ge 0} \|\gamma^n [R(\gamma, A)]^n\| \|f\| \\
&\le \lim_{\gamma\to\infty} \sup_{n\ge 0} \left(\frac{\gamma}{\gamma-\omega}\right)^n \mathcal{B}\|f\| \\
&\le \mathcal{B}\|f\| \\
&\Longrightarrow \\
\|f\| &\le |f| \le \mathcal{B}\|f\|, \quad \forall f \in X.
\end{aligned}
$$

Since $\|T(t)\| \le \mathcal{B}e^{\omega t}$ and $\|[R(\gamma, A)]^n\| \le \frac{\mathcal{B}}{(\gamma-\omega)^n}$ for $\gamma > \omega$ by Corollary 1.70 and (1-56), $|T(t)| \le e^{\omega t}$ and $|R(\gamma, A)| \le \frac{1}{\gamma-\omega}$ for $\gamma > \omega$. Thus, for $\gamma > \omega + |\mathcal{G}|$ the bounded linear operator $\mathcal{G}R(\gamma, A)$ satisfies $|\mathcal{G}R(\gamma, A)| < 1$ and therefore $I - \mathcal{G}R(\gamma, A)$ is invertible. Thus, for $\gamma > \omega + |\mathcal{G}|$

$$
\begin{aligned}
(\gamma I - A - \mathcal{G}) &= \left[I - \mathcal{G}(\gamma I - A)^{-1}\right](\gamma I - A) \\
&= [I - \mathcal{G}R(\gamma, A)](\gamma I - A) \\
&\Longrightarrow \\
(\gamma I - A - \mathcal{G})^{-1} &= R(\gamma, A)[I - \mathcal{G}R(\gamma, A)]^{-1} \\
&= R(\gamma, A)\sum_{k=0}^{\infty}[\mathcal{G}R(\gamma, A)]^k \\
&\Longrightarrow \\
R(\gamma, A + \mathcal{G}) &= R(\gamma, A)\sum_{k=0}^{\infty}[\mathcal{G}R(\gamma, A)]^k. \qquad (1\text{-}70)
\end{aligned}
$$

Which implies for $\gamma > \omega + |\mathcal{G}|$,

$$
\begin{aligned}
\left|(\gamma I - A - \mathcal{G})^{-1}\right| &= \left|R(\gamma, A)\sum_{k=0}^{\infty}[\mathcal{G}R(\gamma, A)]^k\right| \\
&\le |R(\gamma, A)| \sum_{k=0}^{\infty} |\mathcal{G}R(\gamma, A)|^k \\
&\le \frac{1}{\gamma-\omega}\frac{1}{[1 - |\mathcal{G}R(\gamma, A)|]} \\
&= \frac{1}{\gamma - \omega - (\gamma-\omega)|\mathcal{G}R(\gamma, A)|} \\
&\le \frac{1}{\gamma - \omega - (\gamma-\omega)|\mathcal{G}||R(\gamma, A)|} \\
&\le \frac{1}{\gamma-\omega-|\mathcal{G}|} = \frac{1}{\gamma - (\omega + |\mathcal{G}|)}.
\end{aligned}
$$

Because $\overline{D(A)} = X$, $\overline{D(A+\mathcal{G})} = \overline{D(A)} = X$ and therefore from Corollary 1.70 it follows that $A + \mathcal{G}$ is the infinitesimal generator of a C_0-semigroup $S(t)$

satisfying $|S(t)| \le e^{(\omega+|\mathcal{G}|)t}$. Returning to the original norm $\|\cdot\|$ on X we have

$$\|S(t)\| \le \mathcal{B}\|S(t)\| \le \mathcal{B}|S(t)| \le \mathcal{B}e^{(\omega+|\mathcal{G}|)t} \le \mathcal{B}e^{(\omega+\mathcal{B}\|\mathcal{G}\|)t}. \qquad \square$$

We are now interested in the relation between the semigroup $T(t)$ generated by A and the semigroup $S(t)$ generated by $A+\mathcal{G}$. To this end we consider the operator $H(s) = T(t-s)S(s)$. For $f \in D(A) = D(A+\mathcal{G})$, $s \to H(s)f$ is differentiable and $\frac{d}{ds}H(s)f = T(t-s)\mathcal{G}S(s)f$. Integrating $\frac{d}{ds}H(s)f$ from 0 to t by Theorem 1.65 yields, for $f \in D(A)$,

$$\begin{aligned}
\frac{d}{ds}H(s)f &= \frac{\partial}{\partial s}T(t-s)S(s)f \\
&= -T(t-s)AS(s)f + T(t-s)S(s)(A+\mathcal{G})f \\
&= -T(t-s)AS(s)f + T(t-s)(A+\mathcal{G})S(s)f \\
&= -T(t-s)AS(s)f + T(t-s)AS(s)f \\
&\quad + T(t-s)\mathcal{G}S(s)f \\
&= T(t-s)\mathcal{G}S(s)f \\
&\Longrightarrow \\
\int_0^t \frac{d}{ds}H(s)f ds &= \int_0^\infty T(t-s)\mathcal{G}S(s)f ds \\
&\Longrightarrow \\
H(s)f\Big|_0^t &= \int_0^\infty T(t-s)\mathcal{G}S(s)f ds \\
&\Longrightarrow \\
H(t)f - H(0)f &= \int_0^\infty T(t-s)\mathcal{G}S(s)f ds \\
&\Longrightarrow \\
S(t)f &= T(t)f + \int_0^t T(t-s)\mathcal{G}S(s)f ds. \qquad (1\text{-}71)
\end{aligned}$$

Since the operators on both sides of (1-71) are bounded, (1-71) holds for every $f \in X$. The semigroup $S(t)$ is therefore a solution of the integral equation (1-71).

Theorem 1.81. *Let A be a densely defined linear operator with a nonempty resolvent set $\rho(A)$. The abstract Cauchy problem* (1-31) (*see Definition* 1.62) *has a unique solution $u(t)$, which is continuously differentiable on $[0,\infty)$ for every initial value $u_0 \in D(A)$ if and only if A is the infinitesimal generator of a C_0-semigroup $T(t)$.*

Proof. If A is the infinitesimal generator of a C_0-semigroup $T(t)$, then from Theorem 1.65 it follows that for every $u_0 \in D(A)$, $T(t)u_0$ is the unique solution of the abstract Cauchy problem (1-31) with the initial value $u_0 \in D(A)$. Moreover, $T(t)u_0$ is continuously differentiable for $[0,\infty)$.

On the other hand, if the abstract Cauchy problem (1-31) has a unique continuously differentiable solution on $[0,\infty)$ for every initial data $u_0 \in D(A)$, then we will see that A is the infinitesimal generator of a C_0-semigroup $T(t)$. We now assume that for every $u_0 \in D(A)$ the initial value problem (1-31) has a unique continuously differentiable solution on $[0,\infty)$ which we denote by $u(t,u_0)$.

For $u_0 \in D(A)$ we define the graph norm by

$$|u_0|_G = \|u_0\| + \|Au_0\|.$$

Since $\rho(A) \neq \emptyset$, A is closed and therefore $D(A)$ endowed with graph norm is a Banach space which we denote by $[D(A)]$. Let X_{t_0} be the Banach space of continuous functions from $[0,t_0]$ to $[D(A)]$ with usual supremum norm:

$$\|u\| = \sup_{t\in[0,t_0]} |u(\cdot,t)|, \quad u : [0,t_0] \to [D(A)].$$

We consider the mapping $\mathcal{S} : [D(A)] \to X_{t_0}$ defined by $\mathcal{S}u_0 = u(t,u_0)$ for $0 \leq t \leq t_0$. From the linearity of (1-31) and the uniqueness of the solution it is clear that $\mathcal{S}$ is a linear operator defined on all of $[D(A)]$. In fact, if u and v are the solution of (1-31), i.e.,

$$\begin{cases} \frac{du(t)}{dt} = Au(t) \\ u(0) = u_0 \end{cases}, \quad \begin{cases} \frac{dv(t)}{dt} = Av(t), \\ v(0) = v_0 \end{cases}$$

which imply for all $\alpha,\ \beta \in \mathbb{R}$,

$$\begin{cases} \frac{d[\alpha u(t)+\beta v(t)]}{dt} = A(\alpha u(t) + \beta v(t)), \\ [\alpha u(t) + \beta v(t)]\big|_{t=0} = \alpha u(0) + \beta v(0) = \alpha u_0 + \beta v_0 \end{cases}$$

which means that $\mathcal{S}(\alpha u + \beta v)(t) = \alpha u(t,u_0) + \beta v(t,v_0) = \alpha \mathcal{S}u(t) + \beta \mathcal{S}v(t)$.

The operator $\mathcal{S}$ is closed. Indeed, if $f_n \to u_0$ in $[D(A)]$ and $\mathcal{S}f_n \to v$ in X_{t_0}, then from closedness of A and

$$u(t,f_n) = f_n + \int_0^t Au(\tau,f_n)d\tau$$

it follows that, as $n \to \infty$,

$$v(t) = u_0 + \int_0^t Av(\tau)d\tau$$

which implies that $v(t) = u(t,u_0)$ and $\mathcal{S}$ is closed. Therefore, by the closed graph theorem (see Theorem 1.15) $\mathcal{S}$ is bounded, and

$$\sup_{0\leq t\leq t_0} |u(t,u_0)|_G \leq N|u_0|_G. \tag{1-72}$$

We now define a mapping $T(t) : [D(A)] \to [D(A)]$ by $T(t)u_0 = u(t, u_0)$. From the uniqueness of the solution of (1-31) it follows readily that $T(t)$ has the semigroup property. From (1-72) it follows that for $0 \le t \le t_0$, $T(t)$ is uniformly bounded. This implies that $T(t)$ can be extended by

$$\begin{aligned} T(t)u_0 &= T(t - nt_0)T(nt_0)u_0 \\ &= T(t - nt_0)T(\overbrace{t_0 + t_0 + \cdots + t_0}^{n})u_0 \\ &= T(t - nt_0)\overbrace{T(t_0)T(t_0)\cdots T(t_0)}^{n}u_0 \\ &= T(t - nt_0)(T(t_0))^n u_0 \end{aligned}$$

for $nt_0 \le t \le (n_0 + 1)t$ to a semigroup on $[D(A)]$ satisfying $|T(t)u_0|_G \le \mathcal{B}e^{\omega t}|u_0|_G$.

Next we show that

$$T(t)Ay = AT(t)y, \quad y \in D(A^2). \tag{1-73}$$

Setting

$$v(t) = y + \int_0^t u(\tau, Ay)d\tau \tag{1-74}$$

we have

$$\begin{aligned} \frac{dv(t)}{dt} &= u(t, Ay) = Ay + \int_0^t \frac{d}{d\tau}u(\tau, Ay)d\tau \\ &= Ay + \int_0^t Au(\tau, Ay)d\tau = A\left(y + \int_0^t u(\tau, Ay)d\tau\right) \\ &= Av(t). \end{aligned} \tag{1-75}$$

Since $v(0) = y$ we have by the uniqueness of the solution of (1-31), $v(t) = u(t, y)$ and therefore $Au(t, y) = \frac{dv(t)}{dt} = u(t, Ay)$ which is the same as (1-73).

Now since $D(A)$ is dense in X and by our assumption $\rho(A) \ne \emptyset$ also $D(A^2)$ is dense in X. Let $\gamma_0 \in \rho(A)$, $\gamma_0 \ne 0$, be fixed and let $y \in D(A^2)$. If $f = (\gamma_0 I - A)y$, then by (1-73),

$$\begin{aligned} T(t)f = T(t)(\gamma_0 I - A)y &= \gamma_0 T(t)y - T(t)Ay \\ &= \gamma_0 T(t)y - AT(t)y = (\gamma_0 I - A)T(t)y \end{aligned}$$

and therefore, by (1-72),

$$\|T(t)f\| = \|(\gamma_0 I - A)T(t)y\| \le N|T(t)y|_G \le N_1 e^{\omega t}|y|_G. \tag{1-76}$$

But

$$|y|_G = \|y\| + \|Ay\| = \|R(\gamma_0, A)\| + \|AR(\gamma_0, A)f\| \le N_2\|f\|,$$

which implies

$$\|T(t)f\| \le N_3 e^{\omega t}\|f\|. \tag{1-77}$$

Therefore $T(t)$ can be extended to all of X by continuity. After this extension $T(t)$ becomes a C_0-semigroup on X. To complete the proof we have to show that A is the infinitesimal generator of $T(t)$. Denote by A_1 the infinitesimal generator of $T(t)$. If $f \in D(A)$, then by the definition of $T(t)$ we have $T(t)f = u(t, f)$ and therefore by our assumptions

$$\frac{d}{dt}T(t)f = AT(t)f, \quad t \geq 0$$

which implies in particular that $\frac{d}{dt}T(t)f\Big|_{t=0} = Af$ and therefore $D(A) \subset D(A_1)$ and $A_1\Big|_{D(A)} = A$ (see Definition 1.4).

Let $\Re\gamma > \omega$ and $y \in D(A^2)$. It follows from (1-73) and from $A_1\Big|_{D(A)} = A$, $D(A) \subset D(A_1)$ that

$$e^{-\gamma t}AT(t)y = e^{-\gamma t}T(t)Ay = e^{-\gamma t}T(t)A_1 y. \tag{1-78}$$

Integrating (1-78) from 0 to ∞ yields

$$AR(\gamma, A)y = R(\gamma, A_1)A_1 y. \tag{1-79}$$

But $A_1 R(\gamma, A_1)y = R(\gamma, A_1)A_1 y$ and therefore $AR(\gamma, A_1)y = A_1 R(\gamma, A_1)y$ for every $y \in D(A^2)$. Since $A_1 R(\gamma, A_1)$ are uniformly bounded (see (1-47)), A is closed and $D(A^2)$ is dense in X, it follows that $AR(\gamma, A_1)y = A_1 R(\gamma, A_1)y$ for every $y \in X$. This implies $D(A_1) = \text{Range}\{R(\gamma, A_1)\} \subset D(A)$ and $A\Big|_{D(A_1)} = A_1$. Therefore $A = A_1$. $\square$

1.3 Asymptotic Behavior of the Solution of the Abstract Cauchy Problem

This part we have referenced Arendt et al. [5], Lyubich et al. [84], Engel et al. [27], Nagel [91] and Gupur et al. [65].

The asymptotic behavior of solutions of a differential equation is frequently related to spectral properties of the underlying operator. This is well illustrated by the following classical theorem due to Liapunov.

Let A be an $n \times n$ matrix. Then $\lim\limits_{t\to\infty} u(t) = 0$ for every solution u of the differential equation

$$\frac{du(t)}{dt} = Au(t), \quad t \in [0, \infty)$$

if and only if the spectrum of A lies in the open left half complex plane. In this section we will concern the generalizations of this theorem to infinite dimensions.

Definition 1.82. Let X be a Banach space and $A : D(A) \subset X \to X$ be a closed operator. Then

$$s(A) = \sup\{\Re\gamma \mid \gamma \in \sigma(A)\}$$

is called the spectral bound of A.

Let $\{T(t)\}_{t\geq 0}$ be a C_0-semigroup on X. Then

$$\omega(T) = \inf\left\{\omega \in \mathbb{R} \,\middle|\, \begin{array}{l} \text{there exists } \mathcal{B}_\omega \geq 1 \text{ such that} \\ \|T(t)\| \leq \mathcal{B}_\omega e^{\omega t},\ \forall t \geq 0 \end{array}\right\}$$

is called the growth bound of the C_0-semigroup.

Lemma 1.83. *Let $f : [0, \infty) \to \mathbb{R}$ be bounded on compact intervals and subadditive, i.e., $f(s + t) \leq f(s) + f(t)$ for all $s, t \geq 0$. Then*

$$\inf_{t>0} \frac{f(t)}{t} = \lim_{t\to\infty} \frac{f(t)}{t}$$

exists.

Proof. Fix $t_0 > 0$ and write $t = kt_0 + s$ with $k \in \mathbb{N}, s \in [0, t_0)$. The subadditivity implies

$$\begin{aligned}
\frac{f(t)}{t} &= \frac{f(kt_0 + s)}{kt_0 + s} \leq \frac{f(kt_0) + f(s)}{kt_0 + s} \\
&= \frac{f(\overbrace{t_0 + t_0 + \cdots + t_0}^{k})}{kt_0 + s} + \frac{f(s)}{kt_0 + s} \\
&\leq \frac{\overbrace{f(t_0) + f(t_0) + \cdots + f(t_0)}^{k}}{kt_0 + s} + \frac{f(s)}{kt_0 + s} \\
&= \frac{kf(t_0)}{kt_0 + s} + \frac{f(s)}{kt_0 + s} \leq \frac{kf(t_0)}{kt_0} + \frac{f(s)}{kt_0} \\
&= \frac{f(t_0)}{t_0} + \frac{f(s)}{kt_0}.
\end{aligned}$$

Since $k \to \infty$ if $t \to \infty$, we obtain

$$\limsup_{t\to\infty} \frac{f(t)}{t} \leq \frac{f(t_0)}{t_0}$$

for each $t_0 > 0$ and therefore

$$\begin{gathered}
\limsup_{t\to\infty} \frac{f(t)}{t} \leq \inf_{t>0} \frac{f(t)}{t} \leq \liminf_{t\to\infty} \frac{f(t)}{t} \\
\Longrightarrow \\
\inf_{t>0} \frac{f(t)}{t} = \lim_{t\to\infty} \frac{f(t)}{t}.
\end{gathered}$$

□

Lemma 1.84. *For the spectral bound $s(A)$ of a generator A and for the growth bound $\omega(T)$ of the generated C_0-semigroup $\{T(t)\}_{t\geq 0}$, one has*

$$\begin{aligned}-\infty \leq s(A) \leq \omega(T) &= \inf_{t>0} \frac{1}{t} \ln \|T(t)\| \\ &= \lim_{t\to\infty} \frac{1}{t} \ln \|T(t)\| = \frac{1}{t_0} \ln \mathrm{r}(T(t_0)) < \infty\end{aligned}$$

for each $t_0 > 0$. In particular, $\mathrm{r}(T(t)) = e^{\omega(T)t}$ for all $t \geq 0$.

Proof. Because $T(t) = e^{tA}$, $s(A) \leq \omega(T)$ by Definition 1.82. Since the function

$$t \to f(t) \xlongequal{\text{set}} \ln \|T(t)\|$$

satisfies the assumptions of Lemma 1.83 by the semigroup property, we can define

$$v \xlongequal{\text{set}} \inf_{t>0} \frac{1}{t} \ln \|T(t)\| = \lim_{t\to\infty} \frac{1}{t} \ln \|T(t)\|.$$

From this identity it follows that

$$v \leq \frac{1}{t} \ln \|T(t)\| \Longrightarrow e^{vt} \leq \|T(t)\|$$

for all $t \geq 0$, hence $v \leq \omega(T)$ by Definition 1.82. Now we choose $w > v$. Then there exists $t_0 > 0$ such that

$$\frac{1}{t} \ln \|T(t)\| \leq w$$

for all $t \geq t_0$, hence $\|T(t)\| \leq e^{wt}$ for $t \geq t_0$. On $[0, t_0]$, the norm of $T(t)$ remains bounded, so we find $\beta \geq 1$ such that

$$\|T(t)\| \leq \beta e^{wt}$$

for all $t \geq 0$, i.e., $\omega(T) \leq w$. Since we have already proved that $v \leq \omega(T)$, which implies $\omega(T) = v$. From which together with Theorem 1.27 and continuity of e^{τ} for all $\tau \in [0, \infty)$ we have

$$\begin{aligned}\mathrm{r}(T(t)) &= \lim_{n\to\infty} \|[T(t)]^n\|^{\frac{1}{n}} = \lim_{n\to\infty} e^{\ln \|[T(t)]^n\|^{\frac{1}{n}}} \\ &= \lim_{n\to\infty} e^{\frac{1}{n} \ln \|[T(t)]^n\|} = \lim_{n\to\infty} e^{\frac{1}{n} \ln \|\overbrace{T(t)T(t)\cdots T(t)}^{n}\|} \\ &= \lim_{n\to\infty} e^{\frac{1}{n} \ln \|T(\overbrace{t+t+\cdots+t}^{n})\|} = \lim_{n\to\infty} e^{\frac{1}{n} \ln \|T(nt)\|} \\ &= \lim_{n\to\infty} e^{t\frac{1}{nt} \ln \|T(nt)\|} = e^{t \lim_{n\to\infty} \left(\frac{1}{nt} \ln \|T(nt)\|\right)} \\ &= e^{\omega(T)t}.\end{aligned}$$

□

Definition 1.85. Let X be a Banach space. The essential growth bound of the C_0-semigroup $\{T(t)\}_{t\geq 0}$ with generator A is given by

$$\omega_{\mathrm{ess}}(T) = \omega_{\mathrm{ess}}(A) = \inf_{t>0} \frac{1}{t} \ln \|T(t)\|_{\mathrm{ess}}$$

where

$$\|T\|_{\mathrm{ess}} = \inf\{\|T - \mathcal{K}\| \mid \mathcal{K} \text{ is compact on } X\}.$$

We call the set

$$\sigma_{\mathrm{ess}}(T) = \mathbb{C} \setminus \{\gamma \in \mathbb{C} \mid \gamma I - T \text{ is a Fredholm operator}\}$$

the essential spectrum of the $T(t)$. We define the essential spectral radius $\mathrm{r}_{\mathrm{ess}}(T)$ as

$$\begin{aligned} \mathrm{r}_{\mathrm{ess}}(T) &= \sup\{|\gamma| \mid \gamma \in \sigma_{\mathrm{ess}}(T)\} \\ &= \inf\left\{\gamma > 0 \,\middle|\, \begin{matrix} \eta \in \sigma(T),\ |\eta| > \gamma \text{ is a pole} \\ \text{of finite algebraic multiplicity} \end{matrix}\right\}. \end{aligned}$$

If

$$\lim_{t\to\infty} \inf\{\|T(t) - \mathcal{K}\| \mid \mathcal{K} \in \mathcal{L}(X) \text{ is a compact operator}\} = 0,$$

then we say $T(t)$ is a quasi-compact C_0-semigroup.

It is the same as Lemma 1.84 (for the detail proof see Engel et al. [27]) we deduce:

Proposition 1.86. *Let $\{T(t)\}_{t\geq 0}$ be a C_0-semigroup on the Banach space X with generator A. Then*

$$\begin{aligned} -\infty \leq \omega_{\mathrm{ess}}(T) &= \lim_{t\to\infty} \frac{1}{t} \ln \|T(t)\|_{\mathrm{ess}} \\ &= \frac{1}{t_0} \ln \mathrm{r}_{\mathrm{ess}}(T(t_0)) \leq \omega(T) < \infty \end{aligned}$$

for each $t_0 > 0$. Particularly, $\mathrm{r}_{\mathrm{ess}}(T(t)) = e^{t\omega_{\mathrm{ess}}(T)}$ for all $t \geq 0$.

Theorem 1.87. *Let $\{T(t)\}_{t\geq 0}$ be a C_0-semigroup on the Banach space X with generator A. Then*

$$\omega(T) = \max\{\omega_{\mathrm{ess}}(T), s(A)\}.$$

Moreover, for every $\omega > \omega_{\mathrm{ess}}(T)$ the set $\sigma(A) \cap \{\gamma \in \mathbb{C} \mid \Re\gamma \geq \omega\}$ is finite and the corresponding spectral projection has finite rank.

Proof. Proposition 1.86 and Lemma 1.84 give $\omega_{\mathrm{ess}}(T) \leq \omega(T)$ and

$$\omega_{\mathrm{ess}}(T) = \omega(T) \Leftrightarrow \mathrm{r}_{\mathrm{ess}}(T(t)) = e^{t\omega_{\mathrm{ess}}(T)} = e^{t\omega(T)} = \mathrm{r}(T(t))$$

for some $t > 0$. If $\omega_{\text{ess}}(T) < \omega(T)$, then by Nagel [91] there is an eigenvalue γ of $T(t)$ satisfying $|\gamma| = r(T(t)) = e^{t\omega(T)}$, and hence by the spectral mapping theorem (see Engel et al. [27], page 276)

$$\sigma_p(T(t)) \setminus \{0\} = e^{t\sigma_p(A)}, \quad \sigma_r(T(t)) \setminus \{0\} = e^{t\sigma_r(A)}$$

there exists $\tilde{\gamma} \in \sigma_p(A)$ such that $\Re\tilde{\gamma} = \omega(T)$. Thus $\omega_{\text{ess}}(T) < \omega(T)$ implies $s(A) = \omega(T)$.

Assume $\sigma(A) \cap \{\gamma \in \mathbb{C} \mid \Re\gamma \geq \omega\}$ is infinite, then by the spectral inclusion theorem $e^{t\sigma(A)} \subset \sigma(T(t))$ (see Engel et al. [27], page 276) for $t \geq 0$ there exists $t > 0$ such that the set

$$e^{t[\sigma(A)\cap\{\gamma\in\mathbb{C} \mid \Re\gamma\geq\omega\}]} \subset e^{t\sigma(A)} \subset \sigma(T)$$

has an accumulation point s_0. Since $|s_0| \geq e^{t\omega} > e^{t\omega_{\text{ess}}(T)} = \mathrm{r}_{\text{ess}}(T(t))$, which contradicts $s_0 \in \sigma(T)$, and therefore $\sigma(A) \cap \{\gamma \in \mathbb{C} \mid \Re\gamma \geq \omega\}$ is finite. □

By using the following result, which can be found in Nagel [91], we deduce an important result.

Theorem 1.88. *Let $\{T(t)\}_{t\geq 0}$ be a positive semigroup with generator A.*

If $\{T(t)\}_{t\geq 0}$ is bounded and $s(A) = 0$, then the boundary spectrum of $\{T(t)\}_{t\geq 0}$ is imaginary additively cyclic (or simply cyclic), that is,

$$\gamma_1 + i\gamma_2 \in \sigma_b(A) \Longrightarrow \gamma_1 + ik\gamma_2 \in \sigma_b(A), \quad \forall k \in \mathbb{Z},$$

where the boundary spectrum $\sigma_b(A) = \sigma(A) \cap \{\gamma \in \mathbb{C} \mid \Re\gamma = s(A)\}$.

Theorem 1.89. *Let $\{T(t)\}_{t\geq 0}$ be a C_0-semigroup with generator A and take γ_1, γ_2, …, $\gamma_m \in \sigma(A)$ satisfying $\Re\gamma_i > \omega_{\text{ess}}(T)$, $i = 1, 2, \ldots, m$. Then $\gamma_1, \gamma_2, \ldots, \gamma_m$ are isolated spectral values of A with finite algebraic multiplicity. If $\mathbb{P}_1, \mathbb{P}_2, \ldots, \mathbb{P}_m$ denote the corresponding spectral projections and $k_1, k_2, \ldots, k_m$ the corresponding orders of poles of $R(\gamma, A)$, then*

$$T(t) = T_1(t) + T_2(t) + \cdots + T_m(t) + R_m(t),$$

where

$$T_n(t) = e^{\gamma_n t} \sum_{j=0}^{k_n-1} \frac{t^j}{j!}(A - \gamma_n I)^j \mathbb{P}_n, \; n = 1, 2, \ldots, m.$$

Moreover, for every

$$w > \sup\{\omega(T)\} \cup \{\Re\gamma \mid \gamma \in \sigma(A) \setminus \{\gamma_1, \gamma_2, \ldots, \gamma_m\}\}$$

there exists $\beta > 0$ such that

$$\|R_m(t)\| \leq \beta e^{wt}, \quad \forall t \geq 0.$$

Proof. Since $\gamma_i \in \sigma(A)$ and $\Re\gamma_i > \omega_{\text{ess}}(T)$ for all $i = 1, 2, \ldots, m$, by Theorem 1.87 the spectral values $\gamma_1, \gamma_2, \ldots, \gamma_m$ are isolated with finite algebraic multiplicity. Let $\sum\limits_{n=1}^{m} \mathbb{P}_n$ be the spectral projection of A corresponding to the spectral set $\{\gamma_1, \gamma_2, \ldots, \gamma_m\}$, then

$$\begin{aligned} T(t) &= T(t)\left(\sum_{n=1}^{m} \mathbb{P}_n + I - \sum_{n=1}^{m} \mathbb{P}_n\right) \\ &= T(t)\mathbb{P}_1 + T(t)\mathbb{P}_2 + \cdots + T(t)\mathbb{P}_m + T(t)\left(I - \sum_{n=1}^{m} \mathbb{P}_n\right) \\ &= T(t)\mathbb{P}_1 + T(t)\mathbb{P}_2 + \cdots + T(t)\mathbb{P}_m + R_m(t) \end{aligned}$$

and the restricted semigroup $\left\{T(t)\big|_{R(\mathbb{P}_n)}\right\}_{t\geq 0}$ has generator $A\big|_{R(\mathbb{P}_n)}$. Since by Definition 1.23 and Remark 1.24 $R(\mathbb{P}_n)$ is finite dimensional and $(A-\gamma_n I)^{k_n}\Big|_{R(\mathbb{P}_n)} = 0$, we obtain

$$\begin{aligned} T(t)\Big|_{R(\mathbb{P}_n)} &= e^{tA}\Big|_{R(\mathbb{P}_n)} = e^{\gamma_n t + (A-\gamma_n I)t}\Big|_{R(\mathbb{P}_n)} = e^{\gamma_n t} e^{(A-\gamma I)t}\Big|_{R(\mathbb{P}_n)} \\ &= e^{\gamma_n t} \sum_{j=0}^{k_n - 1} \frac{t^j}{j!} (A - \gamma_n I)^j \Big|_{R(\mathbb{P}_n)}, \end{aligned}$$

that is,

$$T_n(t) = T(t)\mathbb{P}_n = e^{\gamma_n t} \sum_{j=0}^{k_n - 1} \frac{t^j}{j!} (A - \gamma_n I)^j \mathbb{P}_n, \ n = 1, 2, \ldots, m.$$

Moreover, by Theorem 1.87,

$$\omega\left(T(t)\Big|_{\ker\left(\sum\limits_{n=1}^{m} \mathbb{P}_n\right)}\right) = \max\left\{\omega_{ess}\left(T(t)\Big|_{\ker \sum\limits_{n=1}^{m} \mathbb{P}_n}\right), s\left(A\Big|_{\ker \sum\limits_{n=1}^{m} \mathbb{P}_n}\right)\right\}.$$

Since

$$\omega\left(T(t)\Big|_{\ker \sum\limits_{n=1}^{m} \mathbb{P}_n}\right) = \omega_{\text{ess}}(T(t))$$

and

$$s\left(A\Big|_{\ker \sum\limits_{n=1}^{m} \mathbb{P}_n}\right) = \sup\{\Re\gamma \mid \gamma \in \sigma(A) \setminus \{\gamma_1, \gamma_2, \ldots, \gamma_m\}\},$$

we have

$$\omega\left(T(t)\Big|_{\ker \sum\limits_{n=1}^{m} \mathbb{P}_n}\right) = \sup\{\omega_{\text{ess}}(T(t))\} \cup \{\Re\gamma \mid \gamma \in \sigma(A) \setminus \{\gamma_1, \gamma_2, \ldots, \gamma_m\}\}.$$

Hence, by Lemma 1.84 for every $w > \omega\left(T(t)\Big|_{\ker \sum_{n=1}^{m} \mathbb{P}_n}\right)$ there exists $\beta > 0$ such that

$$\begin{aligned}\|R_m(t)\| &= \left\| T(t)\left(I - \sum_{n=1}^{m} \mathbb{P}_n\right)\right\| \\ &\leq \left\| T(t)\Big|_{\ker \sum_{n=1}^{m} \mathbb{P}_n}\right\| \left\| I - \sum_{n=1}^{m} \mathbb{P}_n\right\| \\ &\leq \beta e^{wt}, \quad \forall t \geq 0. \qquad \square\end{aligned}$$

Theorem 1.90. *Let $\{T(t)\}_{t\geq 0}$ be a positive semigroup on a Banach lattice X which is uniformly bounded, quasi-compact and has spectral bound zero (i.e., $s(A) = 0$). Then there exists a positive projection $\mathbb{P}$ of finite rank and suitable constants $\delta > 0$, $\mathcal{M} \geq 1$ such that*

$$\|T(t) - \mathbb{P}\| \leq \mathcal{M}e^{-\delta t}, \quad \forall t \geq 0$$

Proof. By Theorem 1.87 the set $\{\gamma \in \sigma(A) \mid \Re\gamma = 0\}$ is finite and by Theorem 1.88 imaginary additively cyclic. Thus it contains only the value $s(A) = 0$, i.e., $\{\gamma \in \sigma(A) \mid \Re\gamma = 0\} = \{0\}$. Then by Theorem 1.89 we have

$$T(t) = \sum_{j=0}^{k-1} \frac{t^j}{j!} A^j \mathbb{P} + R(t), \quad \forall t \geq 0$$

where $\mathbb{P} = \frac{1}{2\pi i}\int_{\overline{\Gamma}}(\gamma I - A)^{-1}d\gamma$, $\overline{\Gamma}$ is a circle with sufficiently small radius and center 0, k is the pole order and $\|R(t)\| \leq \mathcal{M}e^{-\delta t}$ for suitable constants $\delta > 0$ and $\mathcal{M} \geq 1$. Since we assumed that $T(t)$ is uniformly bounded, the pole order k has to be 1 and therefore

$$T(t) = \mathbb{P} + R(t) \Longrightarrow \|T(t) - \mathbb{P}\| = \|R(t)\| \leq \mathcal{M}e^{-\delta t}, \quad t \geq 0. \qquad \square$$

Definition 1.91. Let $\{T(t)\}_{t\geq 0}$ be a C_0-semigroup on a Banach space X. We say $\{T(t)\}_{t\geq 0}$ is asymptotically stable, if $\lim_{t\to\infty} T(t)f = 0$ for all $f \in X$.

Our aim is to find spectral conditions on A which imply the stability of $\{T(t)\}_{t\geq 0}$. There are two features which differ greatly from the finite-dimensional case.

(1) If $T(t)$ is asymptotically stable, then A has no eigenvalues on the imaginary axis, but it can happen that $\sigma(A) \cap i\mathbb{R} \neq \emptyset$.
(2) The spectral mapping theorem does not hold in general. In particular, it can happen that $\Re\gamma < 0$ for all $\gamma \in \sigma(A)$, but $T(t)$ is unbounded.

Lemma 1.92 (Arendt and Batty [5]). *If $T(t)$ is uniformly bounded, then $\sigma_p(A) \cap iR \subset \sigma_r(A)$.*

Theorem 1.93 (ABLV, [5, 84]). *Let* $\{T(t)\}_{t\geq 0}$ *be a uniformly bounded* C_0-*semigroup with generator* A. *Assume that* $\sigma_r(A) \cap i\mathbb{R} = \emptyset$. *If* $\sigma(A) \cap i\mathbb{R}$ *is countable, then* $T(t)$ *is asymptotically stable.*

Definition 1.94. If $Au = 0$ in the abstract Cauchy problem (1-31) has a nonzero solution which satisfies the probability condition, then we call the solution a steady-state (static) solution of the abstract Cauchy problem (1-31).

Remark 1.95. All authors who study queueing theory and reliability theory always add the probability condition to make the steady-state solution unique, see [1], [7], [12], [15], [16], [18], [36], [40], [80] and [109] for instance. If we do not use the probability condition, then the steady-state solution is any one of nonzero solutions of $Au = 0$, that is to say, the eigenvectors corresponding to zero are called steady-state (static) solutions, which was given as a definition in Gupur et al. [65]. In fact, we can make it unique under certain conditions during study of a concrete problem where the geometric multiplicity of zero is 1. The following theorem is just a typical example.

If the geometric multiplicity of zero is not equal to 1, then the probability condition can not make it unique (see Gupur [60]). It is worth studying asymptotic behavior of the time-dependent solution of the abstract Cauchy problem (1-31) in this case.

From the ABLV theorem we deduce an important result about asymptotic behavior of the time-dependent solution of the abstract Cauchy problem (1-31), which was given in Gupur et al. [65].

Theorem 1.96. *Let* X *be a Banach space and* $T(t)$ *be a uniformly bounded* C_0-*semigroup. Suppose that* $\sigma_p(A) \cap i\mathbb{R} = \sigma_p(A^*) \cap i\mathbb{R} = \{0\}$ *and* $\{\gamma \in \mathbb{C} \mid \gamma = ia,\ a \neq 0,\ a \in \mathbb{R}\}$ *belongs to the resolvent set of* A. *If* 0 *is an eigenvalue of* A^* *with algebraic multiplicity* 1, *then the time-dependent solution of the abstract Cauchy problem* (1-31) *strongly converges to its steady-state* (*static*) *solution, that is,*

$$\lim_{t\to\infty} \|T(t)u_0 - \langle q^*, u_0\rangle p\| = 0,$$

where p *and* q^* *satisfy* $Ap = 0$, $A^*q^* = 0$ *and* $\langle p, q^*\rangle = 1$.

Chapter 2

Statement of the Problems

In this chapter, first we introduce the history of Reliability Theory, then we state the history of Mathematical Theory of Reliability, next we introduce Definition of Reliability and Related Concepts, last we introduce Supplementary Variable Technique and put forward the problems that we will research. We mainly refer in this part to Amstadter [3], Cao and Cheng [12], Frankel [34], Gertsbakh [38], Barlow and Proschan [7], Yamada and Osaki [111], Osaki [92].

2.1 Brief Introduction to Reliability Theory

People have long been concerned with reliability of the products they use and of the friends and associates with whom they are in contact. Although the term "reliable" may not have been used specifically, its meaning was intended. The familiar complaint "things do not last as long as they used to do" is a comparison, although a subjective one, of past and present reliability. When we say that someone is reliable, we mean that the person can be depended on to complete a task satisfactorily on time. These description of reliability are qualitative, and they do not involve numerical measures.

Definition 2.1. Reliability is the probability that a device will operate adequately for a given period of time in its intended application.

Variations have been defined for single operation items such as explosive devices and for characteristics which are not time dependent, but essentially this definition applies. The definition includes the term probability, which indicates the use of a quantitative measure. Probability is the likelihood of occurrence of particular form of any event. It can be determined for any of the innumerable consumer or military equipment which are of interest. Only the methods of measuring the probability differ for the various types of equipment.

In addition to the probabilistic aspect, the reliability definition involves three other considerations: satisfactory operation, length of time, and intended applica-

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tion. There must be a definition of what constitutes satisfactory operation. Certainly, equipment does not necessarily have to be totally inoperative for it to be unsatisfactory. If the compression in two cylinders of an automotive engine is low, the performance of the engine will be less than satisfactory. On the other hand, 100 percent compliance of all desired characteristics may not be a realistic definition of satisfactory performance and something less than 100 percent may be acceptable. The United States has had many manned space flights which were considered very successful even though not every item of equipment performed perfectly. Just what constitutes satisfactory performance must be defined if a measure of reliability is to be meaningful.

The length of time of operation is more definitive. A mission is defined as covering some specific length of time. A warranty is written for a specified number of months or years. Once the criteria of satisfactory performance have been defined, the operation of the equipment can be compared with the criteria for the required time period. Even in this area, however, there may be some flexibility. The criteria of acceptability may change as a function of time so that what is considered satisfactory at the end of the operating period may be something less than what was satisfactory at the beginning. A new automobile should not use any oil between oil changes, while the addition of 1 quart of oil every 1000 miles may very well be acceptable for a 5-year old car.

The last consideration – intended application – must also be a part of the reliability definition. Equipment is designed to operate in a given manner under particular sets of conditions. These include environmental conditions and operating conditions which will be encountered in manufacturing, transportation, storage and use. If the equipment fails or degrades excessively when operated in its intended environment, it is unsatisfactory, whereas if it is subjected to stresses in excess of those for which it was designed, failures or degradation may not be reasonable measures of unreliability.

The importance of obtaining highly reliable systems and components has been recognized in recent years. From a purely economic viewpoint, high reliability is desirable to reduce overall costs. The disturbing fact that the yearly cost of maintaining some military systems in an operable state has been as high as ten times the original cost of the equipment emphasizes this need. The failure of a part or component results in the loss of the failed item but most often the old adage about the loss of a horseshoe nail is truly applicable. A leaky brake cylinder can result in a costly repair bill if it causes an accident. A space satellite may be rendered completely useless if a switch fails to operate or a telemetry system becomes inoperative.

Safety is an equally important consideration. A leaky brake cylinder could result in serious personal injury as well as creating undue expense. The collapse of a landing gear on an aircraft could result in the loss of the plane although no passengers were injured. However, the consequences could easily have been much more serious.

Also caused by reliability (or unreliability) are schedule delays, inconvenience, customer dissatisfaction, loss of prestige (possibly on a national level), and, more serious, loss national security. These conditions also involve cost and safety factors. Cost, for example, is inherent in every failure, as is inconvenience or delay. Most failures also involve at least one of the other considerations. Even the prosaic example of a defective television component involves cost, inconvenience, loss of prestige (of the manufacturer or previous serviceman), and customer dissatisfaction.

The need for and importance of reliability have been reflected in the constantly increasing emphasis placed on reliability by both the government and commercial industry. Most department of Defence, NASA, and AEC contracts impose some degree of reliability requirements on the contractor. These range from the definition of system reliability gaols to requirements for actual demonstration of achievement.

The growth, recognition, and definitization of the reliability function were given much impetus during the 1960s. Reliability has become a recognized engineering discipline, with its own methods, procedures, and techniques. In arriving at this status, it encountered growing pains similar to those that quality assurance experienced in the four previous decades. Convincing corporate management that reliability was economically desirable sometimes required an effort comparable to that expended on the performance of the reliability tasks themselves. Hence, the development of reliability in the area of management and control included justification of its existence as well as application of engineering principles to the organization and direction of reliability activities.

During the growth process, three main technical areas of reliability evolved: (1) reliability engineering, covering systems reliability analysis, design review, and related tasks; (2) operations analysis, including failure investigation and corrective action; (3) reliability mathematics. Each of these areas developed its own body of knowledge and, although specific demarcations can not be drawn between one activity and another, in actual practice the reliability functions are often organized into these divisions. Some activities relate to the design organization; e.g., responsibility for design review is sometimes delegated to the design organization itself. In these instances, reliability is then usually responsible for monitoring the management or contractual directives. However, the third function – reliability mathematics – is seldom delegated outside the reliability organization. First, the methods, although not unique to the reliability function, are not familiar to most personnel in design, testing, and other organizational entities; and second, when the term "reliability" is mentioned, the mathematical aspects are the ones usually thought of. In fact, its primary definition is given in mathematical terms involving probability.

Design aspects of reliability cover such functions as system design analyses, comparison of alternate configurations, drawing and specification reviews, compilation of preferred parts and materials lists, and the preparation and analysis of test programs. Some of the specific activities include failure-modes-and-effects analyses, completion of design review checklists, and special studies and investigations.

Environmental studies are frequently included in the reliability design function, as are supplier reliability evaluations. In organizations associated with complex systems, e.g., refineries or spacecraft, the design reliability functions might be separated into systems concepts, mechanical design, and electrical design groups. A fourth group could include such functions as supplier reliability and parts evaluation. Regardless of the particular organizational structure, however, almost all the individual activities make some use of numerical procedures.

The operations reliability functions relate to manufacturing and assembly operations, test performance, failure analysis and corrective action, operating time and cycle data, field operations reports, and other activities associated with the implementation and test of the design. Personnel in this function help to ensure that the design intent is carried out and they report discrepancies in operations and procedures as well as performance anomalies. The operations reliability group provides much of the data on actual equipment reliability that is used by the other reliability groups.

The statistical group is usually the smallest but can provide equal benefits to the overall program. In addition to accomplishing the reliability numerical activities of prediction, apportionment, and assessment, this group (or individual) provides support to the reliability design and operations groups and directly to the design and test engineers. Statistical designs of experiments, goodness-of-fit tests, system prediction techniques, and other mathematical methods are developed and applied to engineering problems. The methods and procedures discussed herein are applicable to both classes of activities, and it is hoped that reliability design and operations personnel and members of engineering organizations as well as reliability statisticians find them informative and useful.

2.2 Brief Introduction to the Mathematical Theory of Reliability

The mathematical theory of reliability has grown out of the demands of modern technology and particularly out of experiences in World War II with complex military systems. One of the first areas of reliability to be approached with any mathematical sophistication was the area of machine maintenance, see Khintchine [73], Palm [94]. The techniques used to solve these problems grew out of the successful experiences of A.K. Erlang [28], C. Palm [94], and others in solving telephone trunking problems. The earliest attempts to justify the Poisson distribution as the input distribution of calls to a telephone trunk also laid the basis for using the exponential as the failure law of complex equipment.

Applications of renewal theory to replacement problems were discussed as early as 1939 by A.J. Lotka [83], who also summarized earlier work in this area. W. Feller [32, 33] is generally credited with developing renewal theory as a mathematical discipline.

In the late 1930s the subject of fatigue life in materials and the related subject of extreme value theory were being studied by Weibull [105], Gumbel [46] and Epstein [29].

During the 1940s the major statistical effort on reliability problems was in the area of quality control, see Duncan [25].

In the early 1950s certain areas of reliability, especially life testing and electronic and missile reliability problems, started to receive a great deal of attention both from mathematical statisticians and from engineers in the military-industrial complex. Among the first groups to face up seriously to the problem of tube reliability were the commercial airlines, see Carhart [11]. Accordingly, the airlines set up an organization called Aeronautical Radio Inc. (ARINC) which collected and analyzed defective tubes and returned them to the tube manufacturer. In its years of operation with the airlines, ARINC achieved notable success in improving the reliability of a number of tube types. The ARINC program since 1950 has been focused on military reliability problems.

In December 1950 the U S Air Force formed an ad hoc Group on reliability of Electronic Equipment to study the whole question of reliability of equipment and to reduce maintenance costs. By late 1952 the Department of Defense (USA) had established the Advisory Group on Reliability of Electronic Equipment (AGREE). AGREE published its first report on reliability in June of 1957. This report included minimum acceptability limits, requirements for reliability tests, effect of storage on reliability, etc. In 1951 Epstein and Sobel [30] began to work in the field of life testing which was to result in a long stream of important and extremely influential papers. This work marked the beginning of the widespread assumption of the distribution in life-testing research.

In the missile industry Richard R. Carhart [11], Buehler [10], Steck [101], Rosenblatt [98], Madansky [85] were also active at this time in promoting interest in reliability and stating the problems of most interest to their technology.

The mathematically important paper of Moore and Shannon [89] appeared in 1956. This was concerned with relay network reliability. Moore and Shannon [89] were stimulated by von Neumann's attempt to describe certain operations of the human brain and the high reliability that has been attained by complex biological organisms.

Largely motivated by vibration problems encountered in the new commercial jet aircraft, Birnbaum and Saunders [9] in 1958 presented an ingenious statistical model of lifetimes of structures under dynamic loading. Their model made it possible to express the probability distribution of life length in terms of the load given as a function of time and of deterioration occurring in time independently of loading.

2.3 Definitions of Reliability and Related Concepts

In considering various reliability problems, we wish to analyze and calculate certain quantities of interest, designated in the literature by a variety of labels: reliability, availability, efficiency, effectiveness, etc. Even though we do not believe a comprehensive set of definitions is required at this point for understanding the models to follow, it may be of some value to present a unified treatment of the various concepts and quantities involved in the subject of mathematical reliability. Specifically, we shall define mathematically a single generalized quantity which will yield most of the fundamental quantities of reliability theory.

To this end we assume a system whose state at time t is described by $X(t) = (X_1(t), X_2(t), \ldots, X_n(t))$, a vector-valued random variable. For example, $X(t)$ may be the one-dimensional variable taking on the value 1 corresponding to the functioning state and 0 corresponding to the failed state. Alternately, $X(t)$ may be a vector of system parameter values, with each component $X_i(t)$ ranging over an interval of real numbers. $X(t)$, being a random variable, will be governed by a distribution function, $F(x_1, x_2, \ldots, x_n; t)$; explicitly, $F(x_1, x_2, \ldots, x_n; t)$ equals the probability that

$$X_1(t) \leq x_1,\ X_2(t) \leq x_2, \ldots,\ X_n(t) \leq x_n.$$

Now corresponding to any state $\quad = (x_1, x_2, \ldots, x_n)$, there is a gain, or payoff, $g(\quad)$. Thus in the two-state example, just given accruing from being in the functioning state ($x = 1$) might be one unit of value, so that $g(1) = 1$, and the gain from being in the field state ($x = 0$) might be 0, so that $g(0) = 0$. The expected gain $G(t)$ at time t will be the quantity of interest; it may be calculated from

$$\begin{aligned} G(t) &= Eg(X(t)) \\ &= \int\int\cdots\int g(x_1, x_2, \ldots, x_n) dF(x_1, x_2, \ldots, x_n; t). \end{aligned} \tag{2-1}$$

Finally, we may average the expected gain $G(t)$ over an interval of time, $a \leq t \leq b$, with respect to some weight function $W(t)$ to obtain

$$H(a, b) = \int_a^b G(t) dW(t). \tag{2-2}$$

Now we are ready to specialize (2-1) and (2-2) to obtain the various basic quantities arising in reliability theory.

Definition 2.2. Reliability is the probability of a device performing its purpose adequately for the period of time intended under the operating conditions encountered.

Ordinarily "the period of time intended" is $[0, t]$. Let $X(t) = 1$ if the device is performing adequately at time t, 0 otherwise; we assume that adequate performance at time t implies adequate performance during $[0, t]$. Then from (2-1),

$$G(t) = Eg(X(t)) = P\{X(t) = 1\} = \text{probability that the device performs adequately over } [0, t]. \tag{2-3}$$

Thus $G(t)$ is the reliability of the device as defined above. In general, we shall assume that, unless repair or replacement occurs, adequate performance at time t implies adequate performance during $[0, t]$.

Definition 2.3. Pointwise availability is the probability that the system will be able to operate within the tolerances at a given instant of time.

As before we let $X(t) = 1$ if the system is operating within tolerances at time t, 0 otherwise. Also as before $g(1) = 1,\ g(0) = 0$. We do not exclude the possibility of repair or replacement before time t. Then

$$G(t) = Eg(X(t)) = P\{X(t) = 1\} \text{ is the probability that the system is operating within tolerances at time } t. \tag{2-4}$$

Thus $G(t)$ now yields pointwise availability at the time t.

Definition 2.4. Interval availability is the expected fraction of a given interval of time that the system will be able to operate within the tolerances (Repair and /or replacement is permitted).

Suppose the given interval of time is $[a, b]$. Then with X, g defined as above and $W(t) = \frac{(t-a)}{b-a}$, we compute from (2-2),

$$H(a, b) = \frac{1}{b-a} \int_a^b G(t)dt = \frac{1}{b-a} \int_a^b Eg(X(t))dt, \tag{2-5}$$

so that under suitable regularity conditions

$$H(a, b) = E \frac{\int_a^b g(X(t))dx}{b-a} \tag{2-6}$$

which is the expected fraction of the time interval $[a, b]$ that the system is operating within tolerances. Thus $H(a, b)$ is the interval availability for the interval $[a.b]$.

Definition 2.5. Limiting interval availability is the expected fraction of time in the long run that the system operates satisfactorily.

To obtain limiting interval availability simply compute

$$\lim_{T\to\infty} H(0, T) \tag{2-7}$$

in (2-5) or (2-6), which in some papers is called "limiting efficiency".

Definition 2.6. Interval reliability is the probability that at a specified time, the system is operating and will continue to operate for an interval of duration, say x (Repair and / or replacement is permitted).

To obtain this quantity, let $X(t) = 1$ if the system is operating at time t, 0 otherwise. Then the interval reliability $R(x, T)$ for an interval of duration x starting at time T is given by

$$R(x, T) = P\{X(t) = 1,\ T \leq t \leq T + x\}. \tag{2-8}$$

Limiting interval reliability is simply the limit of $R(x, T)$ as $T \to \infty$, some researchers call it "strategic reliability".

2.4 Supplementary Variable Technique

Throughout the history of reliability theory, large numbers of reliability problems were solved by using reliability models. There are several methods to establish such models. Among them the supplementary variable technique plays an important role. In 1955, D.R. Cox [18] first put forward the "supplementary variable technique" and established the M/G/1 queueing model. After that, the supplementary variable technique was used by many authors to solve a good number of queueing problems (see Chaudhry and Templeton [13]). In the steady-state case, many problems are more readily treated by the supplementary variable technique than by the imbedded Markov chain. In 1963, Gaver [36] first used the supplementary variable technique to study a reliability model. After that, other researchers widely applied this idea to study many reliability problems, see Linton [81], Subramanian and Ravichandran [100], Ohashi and Nishida [93], Goel et al. [43], Chung [15], Dhillon [19], Garg and Goel [35], Gupta and Sharma [52], Kumar et al. [77], Dhillon and Anude [20], Dhillon and Yang [23], Itoi et al. [71], Adachi et al. [1], Yamashiro [113], Murari and Maruthachalam [90], Yamashiro [112], Kodama and Sawa [75], Dhillon and Natesan [22], Goel et al. [44], Goel and Gupta [39], Goel and Gupta [41], Goel and Gupta [40], Goel et al. [42], Gupta and Agarwal [48], Kodama and Sawa [76], Chung [16], Gupta and Agarwal [47], Gupta and Kumar [49], Kodama et al. [74], Gupta and Sharma [52], Gupta et al. [51], Gupta et al. [50], Mokhles and Abo El-Fotouh [88], Wu et al. [106], Liu and Cao [82], Dhillon and Fashandi [21].

In the supplementary variable technique a non-Markovian process in continuous time is made Markovian by inclusion of one or more supplementary variables. But the above mathematical models established by the supplementary variable technique were described by partial differential equations with integral boundary conditions. So, it is necessary to study their well-posedness. In addition, many of the researchers who established the above reliability models were interested in steady-state reliability indices and therefore they researched steady-state reliability indices under the following hypothesis (see Chung [15], Dhillon [19], Garg and

Goel [35], Gupta and Sharma [52], Kumar et al. [77], Dhillon and Anude [20], Dhillon and Yang [23]):

$$\lim_{t\to\infty} p(x,t) = p(x),$$

here

$$p(x,t) = (p_0(t), p_1(x,t), p_2(x,t), \ldots, p_n(x,t)),$$
$$n \text{ is finite or infinite,}$$
$$p(x) = (p_0, p_1(x), p_2(x), p_3(x), \ldots, p_n(x)),$$
$$n \text{ is finite or infinite.}$$

The above hypothesis means that the time-dependent solutions of the models converge to their steady-state solutions.

There were many researchers who tried to study the time-dependent solutions of the reliability models. Almost all of them first established mathematical models to describe corresponding reliability problems, then studied the time-dependent solution by using the Laplace transform and probability generating functions, next at most determined the expression of the probability generating function's Laplace transform or expression of their time-dependent solutions' Laplace transform if the boundary conditions are simple. Last, they studied steady-state reliability indices such as pointwise availability, operational behavior, and cost analysis (see Itoi et al. [71], Adachi et al. [1], Yamashiro [113], Murari and Maruthachalam [90], Yamashiro [112], Kodama and Sawa [75], Dhillon and Natesan [22], Goel et al. [44], Goel and Gupta [39], Goel and Gupta [41], Goel and Gupta [40], Goel et al. [42], Gupta and Agarwal [48], Kodama and Sawa [76], Chung [16], Gupta and Agarwal [47], Gupta and Kumar [49], Kodama et al. [74], Gupta and Sharma [52], Gupta et al. [51], Gupta et al. [50], Mokhles and Abo El-Fotouh [88], Wu et al. [106], Liu and Cao [82], Dhillon and Fashandi [21]). Roughly speaking, the above researchers obtained existence of the time-dependent solution of the above models and their steady-state reliability indices but have not answered whether the above hypothesis holds. As we know, steady-state solutions of the systems depend on their time-dependent solutions and time-dependent solutions reflect clearly tendency of the systems. Hence, we should study the existence of the time-dependent solutions of the above models and their asymptotic behavior, time-dependent reliability indices and their asymptotic behavior.

In reliability theory literature, solutions to reliability models have been mostly obtained in terms of probability generating functions or Laplace-Stieltjes transform of the probability distributions of interest.This book is an effort to study time-dependent solutions of reliability models, their asymptotic behavior and asymptotic behavior of the reliability indices.

In Chapter 3, we do dynamic analysis for a system which was described by a finite number of partial differential equations with integral boundary conditions. Firstly, we establish the mathematical model by using the supplementary variable technique to describe the system, next by using the knowledge in Chapter 1 we prove that the model has a unique positive time-dependent solution. Thirdly, by using the knowledge in Chapter 1, we prove that the time-dependent solution of the model exponentially converges to its steady-state solution. Lastly, we obtain asymptotic behavior of reliability indices for the system. In Chapter 4, we study a system which was described by an infinite number of partial differential equations with integral boundary conditions. First of all, by using the supplementary variable technique we establish the mathematical model to describe this system, next by using the knowledge in Chapter 1 we obtain the well-posedness of the model. In addition, we prove that the time-dependent solution of this model strongly converges to its steady-state solution and show that our result about convergence is best. Moreover, by using the cone theory we deduce the asymptotic behavior of the reliability indices for the system.

Chapter 3

The System Consisting of a Reliable Machine, an Unreliable Machine and a Storage Buffer with Finite Capacity

In this chapter we discuss a transfer line with reprocess rule, which consists of two machines that are separated by a storage buffer which can store finite workpieces. This transfer line is useful for the computer integrated manufacturing system (CIMS) and therefore several researchers studied special cases of this system, see Shu [99], Tan [103], Gershwin [37], Yeralan [114] for instance. The most general cases were researched by Li and Cao [78]. In 1998, Li and Cao [78] first established a mathematical model of the system by using the supplementary variable technique and studied the steady-state solution and the steady-state reliability indices by using Laplace transform under the following hypothesis:

$$\lim_{t\to\infty} p(x,t) = p(x),$$

here

$$\begin{aligned} p(x,t) &= (p_0(t), p_1(x,t), p_2(x,t)) - \text{time-dependent solution}, \\ p(x) &= (p_0(t), p_1(x), p_2(x)) - \text{steady-state solution}. \end{aligned}$$

But, they did not answer whether the above hypothesis holds. Through reading their paper we find that the above hypothesis in fact implies the following two hypotheses:

Hypothesis 1. The model has a unique time-dependent solution.

Hypothesis 2. The time-dependent solution converges to its steady-state solution.

In this chapter we will study the above two hypotheses.

This chapter is divided into four sections. In Section 1, we first establish the mathematical model of the system consisting of a reliable machine, an unreliable

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machine and a storage buffer which can store at most one piece, then we convert the model into an abstract Cauchy problem by introducing a suitable state space, operators and their domains. In Section 2, by using C_0-semigroup theory we prove that the model has a unique positive time-dependent solution. In Section 3, through discussing spectral properties of the operator corresponding to the model we obtain asymptotic behavior of the time-dependent solution of the model. In Section 4, we deduce asymptotic behavior of the system. In most parts of this chapter we mainly refer to Li et al. [78], Gupur et al. [61, 62, 64] and Zhou et al. [118].

3.1 The Mathematical Model of the System Consisting of a Reliable Machine, an Unreliable Machine and a Storage Buffer Which Can Store at Most One Piece

In this section we discuss a transfer line with reprocess rule, which consists of two machines that are separated by a storage buffer. Workpieces (or parts) enter machine 1 from the outside. Each workpiece is first operated upon in machine 1. After being processed by machine 1, a piece will directly pass to the buffer with probability η_1, otherwise it will be reprocessed by machine 1 with probability $q_1 = 1 - \eta_1$, which is the reprocessed rate of a piece on machine 1. From the buffer, a piece will proceed to machine 2. After being operated on in machine 2, a piece will exit the system with probability η_2, or it will be reprocessed by machine 2 with probability $q_2 = 1 - \eta_2$. Assume the processing time and repair time of machine 1 have constant rates λ and β, and the processing time of machine 2 has general continuous distribution

$$F(t) = 1 - e^{-\int_0^t \mu(x)dx}, \quad t \geq 0.$$

In addition, we assume that the amount of material in storage is 1 at most. This is the number of workpieces in the buffer plus the number (0 or 1) of workpieces in machine 2. There is no workpiece in machine 2 if the buffer becomes empty and machine 2 completes the last remaining workpiece. Moreover, even if a machine is operative, it can not process workpieces if none are available to it or if there is 1 workpiece in the storage containing 1 on machine 2. In the former condition, the machine 2 is said to be starved, in the latter case machine 1 is called blocked. It is assumed that the first machine is never starved and the second machine is never blocked. We also assume the first machine is reliable and each machine processes workpieces one by one. Let $Y(t)$ be the state at time t and $X(t)$ be the elapsed processing time of the piece currently being processed at time t, then $\{Y(t), X(t)\}$ forms a vector Markov process. This system has three states only, they are:

$Y(t) = 0$, at time t machine 1 is processing a piece and there are no pieces in the buffer and on the machine 2.

$Y(t) = 1$, two machines are all processing pieces and there are no pieces in the buffer at time t.

$Y(t) = 2$, two machines are all processing pieces and there is one piece in the buffer at time t. And machine 1 has processed a piece, but the piece still stays on machine 1 because it can not enter the buffer.

From the above definition we have

$$P\left\{\begin{matrix}\text{machine 1 transferred} \\ \text{a piece during } \Delta t\end{matrix}\right\} = \lambda\eta_1\Delta t + o(\Delta t),$$

$$P\left\{\begin{matrix}\text{machine 1 transferred more} \\ \text{than one pieces during } \Delta t\end{matrix}\right\} = o(\Delta t),$$

$$P\left\{\begin{matrix}\text{machine 2 processed} \\ \text{a piece during } \Delta t\end{matrix}\right\} = \mu(x)\Delta t + o(\Delta t),$$

$$P\left\{\begin{matrix}\text{machine 2 processed more} \\ \text{than one pieces during } \Delta t\end{matrix}\right\} = o(\Delta t).$$

If we assume

$$\begin{aligned} p_0(t) &= P\{Y(t) = 0\} \\ &= P\left\{\begin{matrix}\text{machine 1 is processing a piece} \\ \text{and there are no pieces in the} \\ \text{buffer and on the machine 2 at time } t\end{matrix}\right\}, \end{aligned}$$

$$\begin{aligned} p_1(x,t)dx &= P\{Y(t) = 1, x < X(t) \le x + dx\} \\ &= P\left\{\begin{matrix}\text{two machines are all processing} \\ \text{pieces and the elapsed processing} \\ \text{time of the piece on machine 2 lies} \\ \text{in } [x, x+dx) \text{ and there are no} \\ \text{pieces in the buffer at time } t\end{matrix}\right\}, \end{aligned}$$

$$\begin{aligned} p_2(x,t)dx &= P\{Y(t) = 2, x < X(t) \le x + dx\} \\ &= P\left\{\begin{matrix}\text{two machines are all processing pieces} \\ \text{and there is one piece in the buffer,} \\ \text{the elapsed processing time of the piece} \\ \text{on machine 2 lies in } [x, x+dx) \text{ at time } t\end{matrix}\right\}, \end{aligned}$$

then through considering a transition occurring in Δt, and by using the total probability law, conditional probability, Markovian properties and the above formulas we have (for convenience take the state transition Δx and the time transition Δt

as the same Δt)

$$\begin{aligned}
&p_0(t+\Delta t)\\
&= P\{Y(t+\Delta t)=0\}\\
&= P\left\{Y(t+\Delta t)=0, \bigcup_{i=0}^{2} Y(t)=i\right\}\\
&= P\{Y(t+\Delta t)=0, Y(t)=0\}\\
&\quad + P\{Y(t+\Delta t)=0, Y(t)=1\}\\
&\quad + P\{Y(t+\Delta t)=0, Y(t)=2\}\\
&= P\{Y(t+\Delta t)=0 \mid Y(t)=0\}P\{Y(t)=0\}\\
&\quad + P\{Y(t+\Delta t)=0 \mid Y(t)=1\}P\{Y(t)=1\}\\
&\quad + P\{Y(t+\Delta t)=0 \mid Y(t)=2\}P\{Y(t)=2\}\\
&= p_0(t)(1-\lambda\eta_1\Delta t)\\
&\quad + \eta_2\Delta t\int_0^\infty p_1(x,t)\mu(x)dx + o(\Delta t)\\
&\Longrightarrow\\
&\frac{p_0(t+\Delta t)-p_0(t)}{\Delta t}\\
&= -\lambda\eta_1 p_0(t) + \eta_2\int_0^\infty p_1(x,t)\mu(x)dx + \frac{o(\Delta t)}{\Delta t}\\
&\Longrightarrow\\
&\lim_{\Delta t\to 0}\frac{p_0(t+\Delta t)-p_0(t)}{\Delta t}\\
&= -\lim_{\Delta t\to 0}\lambda\eta_1 p_0(t) + \lim_{\Delta t\to 0}\eta_2\int_0^\infty p_1(x,t)\mu(x)dx\\
&\quad + \lim_{\Delta t\to 0}\frac{o(\Delta t)}{\Delta t}\\
&\Longrightarrow\\
&\frac{dp_0(t)}{dt} = -\lambda\eta_1 p_0(t) + \eta_2\int_0^\infty \mu(x)p_1(x,t)dx. \qquad (3\text{-}1)
\end{aligned}$$

$$\begin{aligned}
&p_1(x+\Delta x, t+\Delta t)\\
&= P\left\{\begin{array}{c}\text{two machines are processing pieces and}\\ \text{the elapsed processing time of the piece on}\\ \text{machine 2 lies in } (x, x+dx] \text{ and there are}\\ \text{no pieces in the buffer at time } t\text{, and the}\\ \text{processes are not completed and there are}\\ \text{no pieces in the buffer during } \Delta t\end{array}\right\}\\
&\quad + o(\Delta t)
\end{aligned}$$

$$= P\left\{\begin{array}{c}\text{two machines are processing pieces and}\\ \text{the elapsed processing time of the piece}\\ \text{on machine 2 lies in } (x, x+dx] \text{ and}\\ \text{there are no pieces in the buffer at time } t\end{array}\right\}$$

$$\times P\left\{\begin{array}{c}\text{the process are not completed and there}\\ \text{are no pieces in the buffer during } \Delta t\end{array}\right\}$$

$$+ o(\Delta t)$$

$$= p_1(x,t)[1 - (\lambda\eta_1 + \mu(x))\Delta t] + o(\Delta t)$$

$$\Longrightarrow$$

$$p_1(x+\Delta t, t+\Delta t) - p_1(x,t)$$

$$= -(\lambda\eta_1 + \mu(x))p_1(x,t)\Delta t + o(\Delta t)$$

$$\Longrightarrow$$

$$\frac{p_1(x+\Delta t, t+\Delta t) - p_1(x+\Delta t, t)}{\Delta t} + \frac{p_1(x+\Delta t, t) - p_1(x,t)}{\Delta t}$$

$$= -(\lambda\eta_1 + \mu(x))p_1(x,t) + \frac{o(\Delta t)}{\Delta t}$$

$$\Longrightarrow$$

$$\lim_{\Delta t\to 0}\frac{p_1(x+\Delta t, t+\Delta t) - p_1(x+\Delta t, t)}{\Delta t} + \lim_{\Delta t\to 0}\frac{p_1(x+\Delta t, t) - p_1(x,t)}{\Delta t}$$

$$= -\lim_{\Delta t\to 0}(\lambda\eta_1 + \mu(x))p_1(x,t) + \lim_{\Delta t\to 0}\frac{o(\Delta t)}{\Delta t}$$

$$\Longrightarrow$$

$$\frac{\partial p_1(x,t)}{\partial t} + \frac{\partial p_1(x,t)}{\partial x} = -(\lambda\eta_1 + \mu(x))p_1(x,t). \tag{3-2}$$

$$p_2(x+\Delta x, t+\Delta t)$$

$$= P\left\{\begin{array}{c}\text{two machines are processing pieces,}\\ \text{there are no pieces in the buffer and}\\ \text{the elapsed processing time of the piece}\\ \text{on machine 2 lies in } (x, x+dx]\\ \text{at time } t, \text{ machine 1 transferred}\\ \text{one piece to the buffer during } \Delta t\end{array}\right\}$$

$$+ P\left\{\begin{array}{c}\text{two machines are processing pieces,}\\ \text{there is one piece in the buffer and}\\ \text{the elapsed processing time of the piece}\\ \text{on machine 2 lies in } (x, x+dx] \text{ at time } t,\\ \text{there is one piece in the buffer and}\\ \text{the processes are not completed during } \Delta t\end{array}\right\}$$

$$+ o(\Delta t)$$

$$
\begin{aligned}
&= P\left\{\begin{matrix}\text{two machines are processing pieces,}\\ \text{there are no pieces in the buffer and}\\ \text{the elapsed processing time of the piece}\\ \text{on machine 2 lies in } (x, x+dx] \text{ at time } t\end{matrix}\right\}\\
&\quad \times P\left\{\begin{matrix}\text{machine 1 transferred one}\\ \text{piece to the buffer during } \Delta t\end{matrix}\right\}\\
&\quad + P\left\{\begin{matrix}\text{two machines are processing pieces,}\\ \text{there is one piece in the buffer and}\\ \text{the elapsed processing time of the piece}\\ \text{on machine 2 lies in } (x, x+dx] \text{ at time } t\end{matrix}\right\}\\
&\quad \times P\left\{\begin{matrix}\text{there is one piece in the buffer and}\\ \text{the processes are not completed during } \Delta t\end{matrix}\right\}\\
&\quad + o(\Delta t)\\
&= p_1(x,t)\lambda\eta_1\Delta t + p_2(x,t)[1-\mu(x)\Delta t] + o(\Delta t)\\
&\Longrightarrow\\
&p_2(x+\Delta t, t+\Delta t) - p_2(x,t)\\
&= -\mu(x)p_2(x,t)\Delta t + \lambda\eta_1\Delta t p_1(x,t) + o(\Delta t)\\
&\Longrightarrow\\
&\frac{p_2(x+\Delta t, t+\Delta t) - p_2(x+\Delta t, t)}{\Delta t} + \frac{p_2(x+\Delta t, t) - p_2(x,t)}{\Delta t}\\
&= -\mu(x)p_2(x,t) + \lambda\eta_1 p_1(x,t) + \frac{o(\Delta t)}{\Delta t}\\
&\Longrightarrow\\
&\lim_{\Delta t\to 0}\frac{p_2(x+\Delta t, t+\Delta t) - p_2(x+\Delta t, t)}{\Delta t}\\
&\quad + \lim_{\Delta t\to 0}\frac{p_2(x+\Delta t, t) - p_2(x,t)}{\Delta t}\\
&= -\lim_{\Delta t\to 0}\mu(x)p_2(x,t) + \lim_{\Delta t\to 0}\lambda\eta_1 p_1(x,t) + \lim_{\Delta t\to 0}\frac{o(\Delta t)}{\Delta t}\\
&\Longrightarrow\\
&\frac{\partial p_2(x,t)}{\partial t} + \frac{\partial p_2(x,t)}{\partial x} = -\mu(x)p_2(x,t) + \lambda\eta_1 p_1(x,t). \qquad (3\text{-}3)
\end{aligned}
$$

In the following we discuss boundary conditions and initial conditions.

$$
\begin{aligned}
p_1(0,t) &= P\left\{\begin{matrix}\text{two machines are processing pieces,}\\ \text{there are no pieces in the buffer and}\\ \text{there are no pieces on machine 2 at time } t\end{matrix}\right\}\\
&= P\left\{\begin{matrix}\text{machine 1 is processing a piece,}\\ \text{there are no pieces in the buffer}\\ \text{and on machine 2 at time } t\end{matrix}\right\}
\end{aligned}
$$

$$+ P\left\{\begin{matrix}\text{machine 2 processed unsuccessfully}\\ \text{a piece and there are no pieces in}\\ \text{the buffer at time } t\end{matrix}\right\}$$

$$+ P\left\{\begin{matrix}\text{machine 2 just processed well}\\ \text{one piece and there are no pieces}\\ \text{in the buffer at time } t\end{matrix}\right\}$$

$$= \lambda\eta_1 p_0(t) + q_2 \int_0^\infty \mu(x) p_1(x,t)dx$$

$$+ \eta_2 \int_0^\infty \mu(x) p_2(x,t)dx. \tag{3-4}$$

$$p_2(0,t) = P\left\{\begin{matrix}\text{two machines are processing pieces and}\\ \text{there is one piece in the buffer at time } t\end{matrix}\right\}$$

$$= q_2 \int_0^\infty \mu(x) p_2(x,t)dx. \tag{3-5}$$

If we assume that at time $t = 0$ there are no pieces on machine 1 and machine 2, then

$$p_0(0) = 1, \quad p_1(x,0) = 0, \quad p_2(x,0) = 0. \tag{3-6}$$

By combining (3-1)–(3-3) with (3-4)–(3-6) we obtain the mathematical model describing the above transfer line:

$$\frac{dp_0(t)}{dt} = -\lambda\eta_1 p_0(t) + \eta_2 \int_0^\infty \mu(x) p_1(x,t)dx, \tag{3-7}$$

$$\frac{\partial p_1(x,t)}{\partial t} + \frac{\partial p_1(x,t)}{\partial x} = -(\lambda\eta_1 + \mu(x))p_1(x,t), \tag{3-8}$$

$$\frac{\partial p_2(x,t)}{\partial t} + \frac{\partial p_2(x,t)}{\partial x} = -\mu(x)p_2(x,t) + \lambda\eta_1 p_1(x,t), \tag{3-9}$$

$$p_1(0,t) = \lambda\eta_1 p_0(t) + q_2 \int_0^\infty \mu(x) p_1(x,t)dx$$

$$+ \eta_2 \int_0^\infty \mu(x) p_2(x,t)dx, \tag{3-10}$$

$$p_2(0,t) = q_2 \int_0^\infty \mu(x) p_2(x,t)dx, \tag{3-11}$$

$$p_0(0) = 1, \quad p_i(x,0) = 0, \quad i = 1,2. \tag{3-12}$$

Since $F(t) = 1 - e^{-\int_0^t \mu(x)dx}$ is a probability distribution, $\mu(x)$ satisfies

$$\mu(x) \geq 0, \quad \int_0^\infty \mu(x)dx = \infty. \tag{3-13}$$

Take a state space as follows.

$$X = \left\{ y \;\middle|\; \begin{matrix} y \in \mathbb{R} \times L^1[0,\infty) \times L^1[0,\infty), \\ \|y\| = |y_0| + \|y_1\|_{L^1[0,\infty)} + \|y_2\|_{L^1[0,\infty)} \end{matrix} \right\}.$$

It is obvious that X is a Banach space. In addition, X is a Banach lattice (see Definition 1.10). In fact, we define

$$y \leq p \Longleftrightarrow y_0 \leq p_0,\ y_1(x) \leq p_1(x),\ y_2(x) \leq p_2(x),\ y, p \in X,$$

then $\leq$ is an order in X (see Definition 1.10). Moreover,

$$\begin{aligned}
&|y| = (|y_0|, |y_1(x)|, |y_2(x)|), \quad \forall y \in X,\\
&|y| \leq |p|, \quad y, p \in X\\
&\quad \Longleftrightarrow\\
&|y_0| \leq |p_0|,\ |y_1(x)| \leq |p_1(x)|,\ |y_2(x)| \leq |p_2(x)|,\ y, p \in X.
\end{aligned} \tag{3-14}$$

And $|y| \leq |p|,\ y, p \in X$ implies

$$\begin{aligned}
&|y_0| \leq |p_0|,\ |y_1(x)| \leq |p_1(x)|,\ |y_2(x)| \leq |p_2(x)|\\
&\Longrightarrow\\
&|y_0| \leq |p_0|,\ \int_0^\infty |y_1(x)|dx \leq \int_0^\infty |p_1(x)|dx,\\
&\int_0^\infty |y_2(x)|dx \leq \int_0^\infty |p_2(x)|dx\\
&\Longrightarrow\\
&|y_0| + \int_0^\infty |y_1(x)|dx + \int_0^\infty |y_2(x)|dx\\
&\leq |p_0| + \int_0^\infty |p_1(x)|dx + \int_0^\infty |p_2(x)|dx\\
&\Longrightarrow\\
&\|y\| \leq \|p\|.
\end{aligned}$$

Which means that X is a Banach lattice (see Definition 1.10).

For simplicity, we introduce a notation as follows.

$$\Gamma = \begin{pmatrix} e^{-x} & 0 & 0 \\ \lambda\eta_1 e^{-x} & q_2\mu(x) & \eta_2\mu(x) \\ 0 & 0 & q_2\mu(x) \end{pmatrix}.$$

In the following we define operators and their domains.

$$D(A) = \left\{ y \in X \;\middle|\; \begin{array}{l} \frac{dy_n}{dx} \in L^1[0,\infty),\ y_n(x)\ (n = 1, 2) \text{ are} \\ \text{absolutely continuous functions} \\ \text{and satisfy } y(0) = \int_0^\infty \Gamma y(x)dx \end{array} \right\},$$

$$A\begin{pmatrix} p_0 \\ p_1 \\ p_2 \end{pmatrix}(x) = \begin{pmatrix} -\lambda\eta_1 & 0 & 0 \\ 0 & -\frac{d}{dx} - (\lambda\eta_1 + \mu(x)) & 0 \\ 0 & 0 & -\frac{d}{dx} - \mu(x) \end{pmatrix} \begin{pmatrix} p_0 \\ p_1(x) \\ p_2(x) \end{pmatrix},$$
$$\forall p \in D(A);$$

$$U\begin{pmatrix}p_0\\p_1\\p_2\end{pmatrix}(x)=\begin{pmatrix}0&0&0\\0&0&0\\0&\lambda\eta_1&0\end{pmatrix}\begin{pmatrix}p_0\\p_1(x)\\p_2(x)\end{pmatrix},\quad \forall p\in X;$$

$$E\begin{pmatrix}p_0\\p_1\\p_2\end{pmatrix}(x)=\begin{pmatrix}\eta_2\int_0^\infty \mu(x)p_1(x)dx\\0\\0\end{pmatrix},\quad \forall p\in X.$$

Then the above equations (3-7)–(3-12) can be rewritten as an abstract Cauchy problem in the Banach space X:

$$\begin{cases}\frac{dp(t)}{dt}=(A+U+E)p(t), & t>0,\\ p(0)=(1,0,0).\end{cases}\tag{3-15}$$

3.2 Well-posedness of the System (3-15)

Theorem 3.1. *If $M=\sup\limits_{x\in[0,\infty)}\mu(x)<\infty$, then $A+U+E$ generates a positive contraction C_0-semigroup $T(t)$.*

Proof. The proof of this theorem is split into four steps. Firstly we will prove that $\|(\gamma I-A)^{-1}\|<\frac{1}{\gamma-2M}$ for

$$\gamma>\max\left\{\begin{array}{c}\frac{(Mq_2)^2-(\lambda\eta_1)^2-\lambda\eta_1 M}{M},2M,\\ (Mq_2-\lambda\eta_1)(q_2+2\eta_2)-2M\eta_2\end{array}\right\}.$$

Secondly we will show that $D(A)$ is dense in X. Thus by using the above results and the Hille-Yosida theorem we show that A generates a C_0-semigroup. Thirdly we will verify that U and E are bounded linear operators. From which together with the above steps and the perturbation theorem of C_0-semigroup it follows that $A+U+E$ generates a C_0-semigroup $T(t)$. Lastly we will check that $A+U+E$ is a dispersive operator and therefore will show $T(t)$ is a positive contraction operator.

For any given $y\in X$ we consider the equation $(\gamma I-A)p=y$, that is,

$$(\gamma+\lambda\eta_1)p_0=y_0,\tag{3-16}$$

$$\frac{dp_1}{dx}+(\gamma+\lambda\eta_1+\mu(x))p_1(x)=y_1(x),\tag{3-17}$$

$$\frac{dp_2}{dx}+(\gamma+\mu(x))p_2(x)=y_2(x),\tag{3-18}$$

$$\begin{aligned}p_1(0)=\lambda\eta_1p_0+q_2\int_0^\infty\mu(x)p_1(x)dx\\ +\eta_2\int_0^\infty\mu(x)p_2(x)dx,\end{aligned}\tag{3-19}$$

$$p_2(0) = q_2 \int_0^\infty \mu(x) p_2(x) dx. \tag{3-20}$$

By solving (3-16), (3-17) and (3-18) we have

$$p_0 = \frac{1}{\gamma + \lambda\eta_1} y_0, \tag{3-21}$$

$$\begin{aligned} p_1(x) &= a_1 e^{-(\gamma+\lambda\eta_1)x - \int_0^x \mu(\tau)d\tau} \\ &\quad + e^{-(\gamma+\lambda\eta_1)x - \int_0^x \mu(\tau)d\tau} \\ &\quad \times \int_0^x y_1(\xi) e^{(\gamma+\lambda\eta_1)\xi + \int_0^\xi \mu(\tau)d\tau} d\xi, \end{aligned} \tag{3-22}$$

$$\begin{aligned} p_2(x) &= a_2 e^{-\gamma x - \int_0^x \mu(\tau)d\tau} \\ &\quad + e^{-\gamma x - \int_0^x \mu(\tau)d\tau} \int_0^x y_2(\xi) e^{\gamma\xi + \int_0^\xi \mu(\tau)d\tau} d\xi. \end{aligned} \tag{3-23}$$

Through inserting (3-22) and (3-23) into (3-19) it follows that

$$\begin{aligned} a_1 &= \lambda\eta_1 p_0 + a_1 q_2 \int_0^\infty \mu(x) e^{-(\gamma+\lambda\eta_1)x - \int_0^x \mu(\tau)d\tau} dx \\ &\quad + q_2 \int_0^\infty \mu(x) e^{-(\gamma+\lambda\eta_1)x - \int_0^x \mu(\tau)d\tau} \\ &\quad \times \int_0^x y_1(\xi) e^{(\gamma+\lambda\eta_1)\xi + \int_0^\xi \mu(\tau)d\tau} d\xi dx \\ &\quad + \eta_2 a_2 \int_0^\infty \mu(x) e^{-\gamma x - \int_0^x \mu(\tau)d\tau} dx \\ &\quad + \eta_2 \int_0^\infty \mu(x) e^{-\gamma x - \int_0^x \mu(\tau)d\tau} \int_0^x y_2(\xi) e^{\gamma\xi + \int_0^\xi \mu(\tau)d\tau} d\xi dx \\ &\Longrightarrow \\ &\left[1 - q_2 \int_0^\infty \mu(x) e^{-(\gamma+\lambda\eta_1)x - \int_0^x \mu(\tau)d\tau} dx\right] a_1 \\ &= a_2\eta_2 \int_0^\infty \mu(x) e^{-\gamma x - \int_0^x \mu(\tau)d\tau} dx + \lambda\eta_1 p_0 \\ &\quad + q_2 \int_0^\infty \mu(x) e^{-(\gamma+\lambda\eta_1)x - \int_0^x \mu(\tau)d\tau} \\ &\quad \times \int_0^x y_1(\xi) e^{(\gamma+\lambda\eta_1)\xi + \int_0^\xi \mu(\tau)d\tau} d\xi dx \\ &\quad + \eta_2 \int_0^\infty \mu(x) e^{-\gamma x - \int_0^x \mu(\tau)d\tau} \\ &\quad \times \int_0^x y_2(\xi) e^{\gamma\xi + \int_0^\xi \mu(\tau)d\tau} d\xi dx. \end{aligned} \tag{3-24}$$

By combining (3-20) with (3-23) we deduce

$$
\begin{aligned}
a_2 &= a_2 q_2 \int_0^\infty \mu(x) e^{-\gamma x - \int_0^x \mu(\tau)d\tau} dx \\
&\quad + q_2 \int_0^\infty \mu(x) e^{-\gamma x - \int_0^x \mu(\tau)d\tau} \\
&\quad \times \int_0^x y_2(\xi) e^{\gamma\xi + \int_0^\xi \mu(\tau)d\tau} d\xi dx \\
&\Longrightarrow \\
&\left[1 - q_2 \int_0^\infty \mu(x) e^{-\gamma x - \int_0^x \mu(\tau)d\tau} dx\right] a_2 \\
&= q_2 \int_0^\infty \mu(x) e^{-\gamma x - \int_0^x \mu(\tau)d\tau} \int_0^x y_2(\xi) e^{\gamma\xi + \int_0^\xi \mu(\tau)d\tau} d\xi dx \\
&\Longrightarrow \\
a_2 &= \frac{q_2 \int_0^\infty \mu(x) e^{-\gamma x - \int_0^x \mu(\tau)d\tau} \int_0^x y_2(\xi) e^{\gamma\xi + \int_0^\xi \mu(\tau)d\tau} d\xi dx}{1 - q_2 \int_0^\infty \mu(x) e^{-\gamma x - \int_0^x \mu(\tau)d\tau} dx}.
\end{aligned}
\tag{3-25}
$$

Substituting (3-25) into (3-24) we derive

$$
\begin{aligned}
&\left[1 - q_2 \int_0^\infty \mu(x) e^{-(\gamma+\lambda\eta_1)x - \int_0^x \mu(\tau)d\tau} dx\right] a_1 \\
&= \eta_2 \int_0^\infty \mu(x) e^{-\gamma x - \int_0^x \mu(\tau)d\tau} dx \\
&\quad \times \frac{q_2 \int_0^\infty \mu(x) e^{-\gamma x - \int_0^x \mu(\tau)d\tau} \int_0^x y_2(\xi) e^{\gamma\xi + \int_0^\xi \mu(\tau)d\tau} d\xi dx}{1 - q_2 \int_0^\infty \mu(x) e^{-\gamma x - \int_0^x \mu(\tau)d\tau} dx} \\
&\quad + \lambda\eta_1 p_0 + q_2 \int_0^\infty \mu(x) e^{-(\gamma+\lambda\eta_1)x - \int_0^x \mu(\tau)d\tau} \\
&\quad \times \int_0^x y_1(\xi) e^{(\gamma+\lambda\eta_1)\xi + \int_0^\xi \mu(\tau)d\tau} d\xi dx \\
&\quad + \eta_2 \int_0^\infty \mu(x) e^{-\gamma x - \int_0^x \mu(\tau)d\tau} \int_0^x y_2(\xi) e^{\gamma\xi + \int_0^\xi \mu(\tau)d\tau} d\xi dx \\
&\Longrightarrow \\
a_1 &= \frac{\eta_2 \int_0^\infty \mu(x) e^{-\gamma x - \int_0^x \mu(\tau)d\tau} dx}{1 - q_2 \int_0^\infty \mu(x) e^{-(\gamma+\lambda\eta_1)x - \int_0^x \mu(\tau)d\tau} dx} \\
&\quad \times \frac{q_2 \int_0^\infty \mu(x) e^{-\gamma x - \int_0^x \mu(\tau)d\tau} \int_0^x y_2(\xi) e^{\gamma\xi + \int_0^\xi \mu(\tau)d\tau} d\xi dx}{1 - q_2 \int_0^\infty \mu(x) e^{-\gamma x - \int_0^x \mu(\tau)d\tau} dx} \\
&\quad + \frac{\lambda\eta_1 p_0}{1 - q_2 \int_0^\infty \mu(x) e^{-(\gamma+\lambda\eta_1)x - \int_0^x \mu(\tau)d\tau} dx}
\end{aligned}
$$

$$+\frac{q_2\int_0^\infty \mu(x)e^{-(\gamma+\lambda\eta_1)x-\int_0^x \mu(\tau)d\tau}\int_0^x y_1(\xi)e^{(\gamma+\lambda\eta_1)\xi+\int_0^\xi \mu(\tau)d\tau}d\xi dx}{1-q_2\int_0^\infty \mu(x)e^{-(\gamma+\lambda\eta_1)x-\int_0^x \mu(\tau)d\tau}dx}$$
$$+\frac{\eta_2\int_0^\infty \mu(x)e^{-\gamma x-\int_0^x \mu(\tau)d\tau}\int_0^x y_2(\xi)e^{\gamma\xi+\int_0^\xi \mu(\tau)d\tau}d\xi dx}{1-q_2\int_0^\infty \mu(x)e^{-(\gamma+\lambda\eta_1)x-\int_0^x \mu(\tau)d\tau}dx}. \tag{3-26}$$

From (3-22) and the Fubini theorem we estimate (without loss of generality, assume $\gamma > 0$)

$$\begin{aligned}
\int_0^\infty |p_1(x)|dx \le{}& |a_1|\int_0^\infty e^{-(\gamma+\lambda\eta_1)x-\int_0^x \mu(\tau)d\tau}dx \\
&+\int_0^\infty e^{-(\gamma+\lambda\eta_1)x-\int_0^x \mu(\tau)d\tau} \\
&\times\int_0^x |y_1(\xi)|e^{(\gamma+\lambda\eta_1)\xi+\int_0^\xi \mu(\tau)d\tau}d\xi dx \\
\le{}& |a_1|\int_0^\infty e^{-(\gamma+\lambda\eta_1)x}dx \\
&+\int_0^\infty |y_1(\xi)|e^{(\gamma+\lambda\eta_1)\xi+\int_0^\xi \mu(\tau)d\tau} \\
&\times\int_\xi^\infty e^{-(\gamma+\lambda\eta_1)x-\int_0^x \mu(\xi)d\xi}dxd\xi \\
={}& |a_1|\frac{1}{\gamma+\lambda\eta_1}\left[-e^{-(\gamma+\lambda\eta_1)x}\right]\Big|_0^\infty \\
&+\int_0^\infty |y_1(\xi)|e^{(\gamma+\lambda\eta_1)\xi} \\
&\times\int_\xi^\infty e^{-(\gamma+\lambda\eta_1)x}e^{-\int_0^x \mu(\tau)d\tau+\int_0^\xi \mu(\tau)d\tau}dxd\xi \\
={}& \frac{|a_1|}{\gamma+\lambda\eta_1} \\
&+\int_0^\infty |y_1(\xi)|e^{(\gamma+\lambda\eta_1)\xi} \\
&\quad\times\int_\xi^\infty e^{-(\gamma+\lambda\eta_1)x}e^{-\int_\xi^x \mu(\tau)d\tau}dxd\xi \\
\le{}& \frac{1}{\gamma+\lambda\eta_1}|a_1| \\
&+\int_0^\infty |y_1(\xi)|e^{(\gamma+\lambda\eta_1)\xi}\int_\xi^\infty e^{-(\gamma+\lambda\eta_1)x}dxd\xi \\
={}& \frac{|a_1|}{\gamma+\lambda\eta_1}
\end{aligned}$$

$$
\begin{aligned}
&+\int_0^\infty |y_1(\xi)|e^{(\gamma+\lambda\eta_1)\xi}\frac{-1}{\gamma+\lambda\eta_1}e^{-(\gamma+\lambda\eta_1)x}\Big|_{x=\xi}^{x=\infty}\\
&=\frac{1}{\gamma+\lambda\eta_1}|a_1|+\frac{1}{\gamma+\lambda\eta_1}\int_0^\infty |y_1(\xi)|d\xi\\
&=\frac{1}{\gamma+\lambda\eta_1}|a_1|+\frac{1}{\gamma+\lambda\eta_1}\|y_1\|_{L^1[0,\infty)}.
\end{aligned}\tag{3-27}
$$

In (3-27) we used the following inequalities:

$$
e^{-\int_0^x \mu(\tau)d\tau}\le 1,\ x\ge 0;\quad e^{-\int_\xi^x \mu(\tau)d\tau}\le 1,\ x\ge\xi\ge 0.
$$

From which together with (3-23) and the Fubini theorem we have (without loss of generality, assume $\gamma>0$)

$$
\begin{aligned}
\int_0^\infty |p_2(x)|dx &\le |a_2|\int_0^\infty e^{-\gamma x-\int_0^x \mu(\tau)d\tau}dx\\
&\quad+\int_0^\infty e^{-\gamma x-\int_0^x \mu(\tau)d\tau}\int_0^x |y_2(\xi)|e^{\gamma\xi+\int_0^\xi \mu(\tau)d\tau}d\xi dx\\
&\le |a_2|\int_0^\infty e^{-\gamma x}dx\\
&\quad+\int_0^\infty |y_2(\xi)|e^{\gamma\xi+\int_0^\xi \mu(\tau)d\tau}\int_\xi^\infty e^{-\gamma x-\int_0^x \mu(\tau)d\tau}dxd\xi\\
&=\frac{1}{\gamma}|a_2|+\int_0^\infty |y_2(\xi)|e^{\gamma\xi}\\
&\quad\times\int_\xi^\infty e^{-\gamma x}e^{-\int_0^x \mu(\tau)d\tau+\int_0^\xi \mu(\tau)d\tau}dxd\xi\\
&=\frac{1}{\gamma}|a_2|+\int_0^\infty |y_2(\xi)|e^{\gamma\xi}\int_\xi^\infty e^{-\gamma x}e^{-\int_\xi^x \mu(\tau)d\tau}dxd\xi\\
&\le\frac{1}{\gamma}|a_2|+\int_0^\infty |y_2(\xi)|e^{\gamma\xi}\int_\xi^\infty e^{-\gamma x}dxd\xi\\
&=\frac{1}{\gamma}|a_2|+\int_0^\infty y_2(\xi)e^{\gamma\xi}\left[\frac{-1}{\gamma}e^{-\gamma x}\right]\Big|_{x=\xi}^{x=\infty}\\
&=\frac{1}{\gamma}|a_2|+\frac{1}{\gamma}\|y_2\|_{L^1[0,\infty)}.
\end{aligned}\tag{3-28}
$$

By (3-21), (3-25), (3-26) and the Fubini theorem we calculate, for $\gamma>Mq_2$,

$$
\begin{aligned}
|a_2|&\le\frac{q_2\int_0^\infty \mu(x)e^{-\gamma x-\int_0^x \mu(\tau)d\tau}\int_0^x |y_2(\xi)|e^{\gamma\xi+\int_0^\xi \mu(\tau)d\tau}d\xi dx}{1-q_2\int_0^\infty \mu(x)e^{-\gamma x-\int_0^x \mu(\tau)d\tau}dx}\\
&=\frac{q_2\int_0^\infty |y_2(\xi)|e^{\gamma\xi+\int_0^\xi \mu(\tau)d\tau}\int_\xi^\infty \mu(x)e^{-\gamma x-\int_0^x \mu(\tau)d\tau}dxd\xi}{1-q_2\int_0^\infty \mu(x)e^{-\gamma x-\int_0^x \mu(\tau)d\tau}dx}
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{q_2 \int_0^\infty |y_2(\xi)| e^{\gamma\xi} \int_\xi^\infty \mu(x) e^{-\gamma x} e^{-\int_0^x \mu(\tau)d\tau + \int_0^\xi \mu(\tau)d\tau} dx d\xi}{1 - q_2 \int_0^\infty \mu(x) e^{-\gamma x - \int_0^x \mu(\tau)d\tau} dx} \\
&= \frac{q_2 \int_0^\infty |y_2(\xi)| e^{\gamma\xi} \int_\xi^\infty \mu(x) e^{-\gamma x} e^{-\int_\xi^x \mu(\tau)d\tau} dx d\xi}{1 - q_2 \int_0^\infty \mu(x) e^{-\gamma x - \int_0^x \mu(\tau)d\tau} dx} \\
&\le \frac{M q_2 \int_0^\infty |y_2(\xi)| e^{\gamma\xi} \int_\xi^\infty e^{-\gamma x} dx d\xi}{1 - q_2 M \int_0^\infty e^{-\gamma x} dx} \\
&= \frac{M q_2 \frac{1}{\gamma} \int_0^\infty |y_2(\xi)| d\xi}{1 - \frac{M q_2}{\gamma}} \\
&= \frac{M q_2}{\gamma - M q_2} \|y_2\|_{L^1[0,\infty)}.
\end{aligned} \tag{3-29}
$$

$$
\begin{aligned}
|a_1| \le\ & \frac{\eta_2 \int_0^\infty \mu(x) e^{-\gamma x - \int_0^x \mu(\tau)d\tau} dx}{1 - q_2 \int_0^\infty \mu(x) e^{-(\gamma+\lambda\eta_1)x - \int_0^x \mu(\tau)d\tau} dx} \\
& \times \frac{q_2 \int_0^\infty \mu(x) e^{-\gamma x - \int_0^x \mu(\tau)d\tau} \int_0^x |y_2(\xi)| e^{\gamma\xi + \int_0^\xi \mu(\tau)d\tau} d\xi dx}{1 - q_2 \int_0^\infty \mu(x) e^{-\gamma x - \int_0^x \mu(\tau)d\tau} dx} \\
& + \frac{\lambda\eta_1 |p_0|}{1 - q_2 \int_0^\infty \mu(x) e^{-(\gamma+\lambda\eta_1)x - \int_0^x \mu(\tau)d\tau} dx} \\
& + \frac{q_2 \int_0^\infty \mu(x) e^{-(\gamma+\lambda\eta_1)x - \int_0^x \mu(\tau)d\tau} \int_0^x |y_1(\xi)| e^{(\gamma+\lambda\eta_1)\xi + \int_0^\xi \mu(\tau)d\tau} d\xi dx}{1 - q_2 \int_0^\infty \mu(x) e^{-(\gamma+\lambda\eta_1)x - \int_0^x \mu(\tau)d\tau} dx} \\
& + \frac{\eta_2 \int_0^\infty \mu(x) e^{-\gamma x - \int_0^x \mu(\tau)d\tau} \int_0^x |y_2(\xi)| e^{\gamma\xi + \int_0^\xi \mu(\tau)d\tau} d\xi dx}{1 - q_2 \int_0^\infty \mu(x) e^{-(\gamma+\lambda\eta_1)x - \int_0^x \mu(\tau)d\tau} dx} \\
\le\ & \frac{q_2 \eta_2 \frac{M}{\gamma} \frac{M}{\gamma}}{\left(1 - \frac{M q_2}{\gamma+\lambda\eta_1}\right)\left(1 - \frac{M q_2}{\gamma}\right)} \|y_2\|_{L^1[0,\infty)} + \frac{\lambda\eta_1}{1 - \frac{M q_2}{\gamma+\lambda\eta_1}} |p_0| \\
& + \frac{q_2 \frac{M}{\gamma+\lambda\eta_1}}{1 - \frac{M q_2}{\gamma+\lambda\eta_1}} \|y_1\|_{L^1[0,\infty)} + \frac{\eta_2 \frac{M}{\gamma}}{1 - \frac{M q_2}{\gamma+\lambda\eta_1}} \|y_2\|_{L^1[0,\infty)} \\
=\ & \frac{M^2 q_2 \eta_2 (\gamma+\lambda\eta_1)}{\gamma(\gamma+\lambda\eta_1 - M q_2)(\gamma - M q_2)} \|y_2\|_{L^1[0,\infty)} \\
& + \frac{\lambda\eta_1}{\gamma+\lambda\eta_1 - M q_2} |y_0| + \frac{M q_2}{\gamma+\lambda\eta_1 - M q_2} \|y_1\|_{L^1[0,\infty)} \\
& + \frac{M \eta_2 (\gamma+\lambda\eta_1)}{\gamma(\gamma+\lambda\eta_1 - M q_2)} \|y_2\|_{L^1[0,\infty)} \\
=\ & \frac{M \eta_2 (\gamma+\lambda\eta_1)}{(\gamma+\lambda\eta_1 - M q_2)(\gamma - M q_2)} \|y_2\|_{L^1[0,\infty)}
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{\lambda\eta_1}{\gamma+\lambda\eta_1-Mq_2}|y_0| \\
&+\frac{Mq_2}{\gamma+\lambda\eta_1-Mq_2}\|y_1\|_{L^1[0,\infty)}.
\end{aligned}
\tag{3-30}
$$

In (3-29) and (3-30) we used the following inequalities:

$$
\begin{aligned}
& e^{-\int_\xi^x \mu(\tau)d\tau} \leq 1, \quad \forall x \geq \xi \geq 0. \\
& 1-q_2\int_0^\infty \mu(x)e^{-\gamma x-\int_0^x \mu(\tau)d\tau}dx \\
& \geq 1-Mq_2\int_0^\infty e^{-\gamma x-\int_0^x \mu(\tau)d\tau}dx \\
& \geq 1-Mq_2\int_0^\infty e^{-\gamma x}dx \\
& = 1-\frac{Mq_2}{\gamma}. \\
& 1-q_2\int_0^\infty \mu(x)e^{-(\gamma+\lambda\eta_1)x-\int_0^x \mu(\tau)d\tau}dx \\
& \geq 1-Mq_2\int_0^\infty e^{-(\gamma+\lambda\eta_1)x-\int_0^x \mu(\tau)d\tau}dx \\
& \geq 1-Mq_2\int_0^\infty e^{-(\gamma+\lambda\eta_1)x}dx \\
& \geq 1-\frac{Mq_2}{\gamma+\lambda\eta_1}.
\end{aligned}
$$

If we assume

$$
\gamma > \max\left\{ \begin{array}{l} \frac{(Mq_2)^2-(\lambda\eta_1)^2-\lambda\eta_1 M}{M}, 2M, \\ (Mq_2-\lambda\eta_1)(q_2+2\eta_2)-2M\eta_2 \end{array} \right\},
$$

then by combining (3-27) and (3-28) with (3-29) and (3-30) respectively, and using (3-21) we estimate

$$
\begin{aligned}
\|p\| &= |p_0|+\|p_1\|_{L^1[0,\infty)}+\|p_2\|_{L^1[0,\infty)} \\
&\leq \frac{1}{\gamma+\lambda\eta_1}|y_0| \\
&\quad+\frac{1}{\gamma+\lambda\eta_1}\frac{M\eta_2(\gamma+\lambda\eta_1)}{(\gamma+\lambda\eta_1-Mq_2)(\gamma-Mq_2)}\|y_2\|_{L^1[0,\infty)} \\
&\quad+\frac{\lambda\eta_1}{(\gamma+\lambda\eta_1)(\gamma+\lambda\eta_1-Mq_2)}|y_0| \\
&\quad+\frac{Mq_2}{(\gamma+\lambda\eta_1)(\gamma+\lambda\eta_1-Mq_2)}\|y_1\|_{L^1[0,\infty)}
\end{aligned}
$$

$$
\begin{aligned}
&+ \frac{1}{\gamma+\lambda\eta_1}\|y_1\|_{L^1[0,\infty)} + \frac{Mq_2}{\gamma(\gamma-Mq_2)}\|y_2\|_{L^1[0,\infty)} \\
&+ \frac{1}{\gamma}\|y_2\|_{L^1[0,\infty)} \\
= {} & \frac{\gamma+2\lambda\eta_1-Mq_2}{(\gamma+\lambda\eta_1)(\gamma+\lambda\eta_1-Mq_2)}|y_0| \\
&+ \frac{1}{\gamma+\lambda\eta_1-Mq_2}\|y_1\|_{L^1[0,\infty)} \\
&+ \frac{\gamma+\lambda\eta_1-Mq_2+M\eta_2}{(\gamma+\lambda\eta_1-Mq_2)(\gamma-Mq_2)}\|y_2\|_{L^1[0,\infty)} \\
\le {} & \frac{\gamma+\lambda\eta_1-Mq_2+M\eta_2}{(\gamma+\lambda\eta_1-Mq_2)(\gamma-Mq_2)} \\
&\times \left(|y_0| + \|y_1\|_{L^1[0,\infty)} + \|y_2\|_{L^1[0,\infty)}\right) \\
= {} & \frac{\gamma+\lambda\eta_1-Mq_2+M\eta_2}{(\gamma+\lambda\eta_1-Mq_2)(\gamma-Mq_2)}\|y\| \\
< {} & \frac{1}{\gamma-2M}\|y\|. \qquad (3\text{-}31)
\end{aligned}
$$

In (3-31) we used the following inequalities (noting $q_2+\eta_2=1$):

$$
\begin{aligned}
&\gamma > \frac{(Mq_2)^2-(\lambda\eta_1)^2-\lambda\eta_1 M}{M} \\
&\Longleftrightarrow \\
&M\gamma > (Mq_2)^2-(\lambda\eta_1)^2-\lambda\eta_1 M \\
&\Longleftrightarrow \\
&M(q_2+\eta_2)\gamma > (Mq_2)^2-(\lambda\eta_1)^2-\lambda\eta_1 M(q_2+\eta_2) \\
&\Longleftrightarrow \\
&(Mq_2+M\eta_2)\gamma > -Mq_2\lambda\eta_1+(Mq_2)^2-(\lambda\eta_1)^2-\lambda\eta_1 M\eta_2 \\
&\Longleftrightarrow \\
&-(\lambda\eta_1-Mq_2)\gamma+(\lambda\eta_1+M\eta_2)\gamma \\
&> -Mq_2(\lambda\eta_1-Mq_2)-\lambda\eta_1(\lambda\eta_1+M\eta_2) \\
&\Longleftrightarrow \\
&(\lambda\eta_1+M\eta_2)\gamma+\lambda\eta_1(\lambda\eta_1+M\eta_2) \\
&> (\lambda\eta_1-Mq_2)\gamma-Mq_2(\lambda\eta_1-Mq_2) \\
&\Longleftrightarrow \\
&(\gamma+\lambda\eta_1)(\lambda\eta_1+M\eta_2) > (\lambda\eta_1-Mq_2)(\gamma-Mq_2) \\
&\Longleftrightarrow \\
&(\gamma+\lambda\eta_1)(\gamma-Mq_2)+(\gamma+\lambda\eta_1)(\lambda\eta_1+M\eta_2) \\
&> (\gamma+\lambda\eta_1)(\gamma-Mq_2)+(\lambda\eta_1-Mq_2)(\gamma-Mq_2)
\end{aligned}
$$

$$\Longleftrightarrow$$
$$(\gamma - Mq_2 + \lambda\eta_1 + M\eta_2)(\gamma + \lambda\eta_1)$$
$$> (\gamma + \lambda\eta_1 + \lambda\eta_1 - Mq_2)(\gamma - Mq_2)$$
$$\Longleftrightarrow$$
$$(\gamma + \lambda\eta_1 - Mq_2 + M\eta_2)(\gamma + \lambda\eta_1)$$
$$> (\gamma + 2\lambda\eta_1 - Mq_2)(\gamma - Mq_2)$$
$$\Longleftrightarrow$$
$$\frac{\gamma + \lambda\eta_1 - Mq_2 + M\eta_2}{\gamma - Mq_2} > \frac{\gamma + 2\lambda\eta_1 - Mq_2}{\gamma + \lambda\eta_1}$$
$$\Longleftrightarrow$$
$$\frac{\gamma + \lambda\eta_1 - Mq_2 + M\eta_2}{(\gamma + \lambda\eta_1 - Mq_2)(\gamma - Mq_2)}$$
$$> \frac{\gamma + 2\lambda\eta_1 - Mq_2}{(\gamma + \lambda\eta_1)(\gamma + \lambda\eta_1 - Mq_2)}.$$
$$\gamma > (Mq_2 - \lambda\eta_1)(q_2 + 2\eta_2) - 2M\eta_2$$
$$\xLeftrightarrow{\times M}$$
$$\gamma M > (Mq_2 - \lambda\eta_1)(Mq_2 + 2M\eta_2) - 2M^2\eta_2$$
$$\Longleftrightarrow$$
$$\gamma M > -(\lambda\eta_1 - Mq_2)(Mq_2 + 2M\eta_2) - 2M^2\eta_2$$
$$\Longleftrightarrow$$
$$\gamma(Mq_2 + M\eta_2) + (\lambda\eta_1 - Mq_2)(Mq_2 + 2M\eta_2)$$
$$> -2M^2\eta_2$$
$$\Longleftrightarrow$$
$$\gamma M\eta_2 + \gamma(Mq_2 + M\eta_2) + (\lambda\eta_1 - Mq_2)(Mq_2 + 2M\eta_2)$$
$$> \gamma M\eta_2 - 2M^2\eta_2$$
$$\Longleftrightarrow$$
$$\gamma(Mq_2 + 2M\eta_2) + (\lambda\eta_1 - Mq_2)(Mq_2 + 2M\eta_2)$$
$$> \gamma M\eta_2 - 2M^2\eta_2$$
$$\Longleftrightarrow$$
$$(\gamma + \lambda\eta_1 - Mq_2)(Mq_2 + 2M\eta_2)$$
$$> \gamma M\eta_2 - 2M^2\eta_2$$
$$\Longleftrightarrow$$
$$(\gamma + \lambda\eta_1 - Mq_2)(2Mq_2 + 2M\eta_2 - Mq_2)$$
$$> \gamma M\eta_2 - 2M^2\eta_2$$

$$
\begin{aligned}
&\Longleftrightarrow\\
&-Mq_2(\gamma+\lambda\eta_1-Mq_2)+2M(\eta_2+q_2)(\gamma+\lambda\eta_1-Mq_2)\\
&>\gamma M\eta_2-2M^2\eta_2\\
&\Longleftrightarrow\\
&-Mq_2(\gamma+\lambda\eta_1-Mq_2)+2M(\gamma+\lambda\eta_1-Mq_2)\\
&>\gamma M\eta_2-2M^2\eta_2\\
&\Longleftrightarrow\\
&-\gamma Mq_2-Mq_2(\gamma+\lambda\eta_1-Mq_2)\\
&>-\gamma Mq_2+\gamma M\eta_2-2M(\gamma+\lambda\eta_1-Mq_2)-2M^2\eta_2\\
&\Longleftrightarrow\\
&\gamma(\gamma+\lambda\eta_1)-\gamma Mq_2-Mq_2(\gamma+\lambda\eta_1-Mq_2)\\
&>\gamma(\gamma+\lambda\eta_1)+\gamma(-Mq_2+M\eta_2)\\
&\quad-2M(\gamma+\lambda\eta_1-Mq_2+M\eta_2)\\
&\Longleftrightarrow\\
&\gamma(\gamma+\lambda\eta_1-Mq_2)-Mq_2(\gamma+\lambda\eta_1-Mq_2)\\
&>\gamma(\gamma+\lambda\eta_1-Mq_2+M\eta_2)-2M(\gamma+\lambda\eta_1-Mq_2+M\eta_2)\\
&\Longleftrightarrow\\
&(\gamma-Mq_2)(\gamma+\lambda\eta_1-Mq_2)\\
&>(\gamma+\lambda\eta_1-Mq_2+M\eta_2)(\gamma-2M)\\
&\Longleftrightarrow\\
&\frac{1}{\gamma-2M}>\frac{\gamma+\lambda\eta_1-Mq_2+M\eta_2}{(\gamma-Mq_2)(\gamma+\lambda\eta_1-Mq_2)}.
\end{aligned}
$$

(3-31) shows that

$$(\gamma I-A)^{-1}:X\to D(A),\quad \left\|(\gamma I-A)^{-1}\right\|<\frac{1}{\gamma-2M}$$

as

$$\gamma>\max\left\{\begin{array}{c}\frac{(Mq_2)^2-(\lambda\eta_1)^2-\lambda\eta_1 M}{M},2M,\\ (Mq_2-\lambda\eta_1)(q_2+2\eta_2)-2M\eta_2\end{array}\right\}.$$

In the second step we will prove that $D(A)$ is dense in X. Let

$$\mathbb{L}=\left\{(p_0,p_1,p_2)\;\middle|\;\begin{array}{l}p_i\in C_0^\infty[0,\infty)\text{ and there exist}\\ \text{constants }c_i>0\text{ such that }p_i(x)=0\\ \text{for }x\in[0,c_i],\ i=1,2\end{array}\right\},$$

then by Adams [2] we know that $\mathbb{L}$ is dense in X. Hence, in order to prove that $D(A)$ is dense in X, it suffices to prove $\mathbb{L}\subset\overline{D(A)}$. In fact, if $\mathbb{L}\subset\overline{D(A)}$, then $X=\overline{\mathbb{L}}\subset\overline{\overline{D(A)}}=\overline{D(A)}$ and $D(A)\subset X\Rightarrow\overline{D(A)}\subset\overline{X}=X$ imply $\overline{D(A)}=X$.

Take any $p \in \mathbb{L}$, then there exist constants $c_i > 0$ such that $p_i(x) = 0$ for $x \in [0, c_i]$, $i = 1, 2$. From which we deduce $p_i(x) = 0$ for $x \in [0, 2s]$, where $0 < 2s < \min\{c_1, c_2\}$. Define

$$\begin{aligned} f^s(0) &= (p_0, f_1^s(0), f_2^s(0)) \\ &= \bigg(p_0, \lambda\eta_1 p_0 + q_2 \int_{2s}^{\infty} p_1(x)\mu(x)dx \\ &\quad + \eta_2 \int_{2s}^{\infty} p_2(x)\mu(x)dx, q_2 \int_{2s}^{\infty} p_2(x)\mu(x)dx\bigg), \\ f^s(x) &= (p_0, f_1^s(x), f_2^s(x)), \end{aligned}$$

where

$$f_i^s(x) = \begin{cases} f_i^s(0)\left(1 - \frac{x}{s}\right)^2 & x \in [0, s) \\ -u_i(x-s)^2(x-2s)^2 & x \in [s, 2s) \\ p_i(x) & x \in [2s, \infty) \end{cases}, \quad i = 1, 2.$$

$$u_i = \frac{f_i^s(0)\int_0^s \mu(x)\left(1 - \frac{x}{s}\right)^2 dx}{\int_s^{2s} \mu(x)(x-s)^2(x-2s)^2 dx}, \qquad i = 1, 2.$$

It is easy to check $f^s \in D(A)$. Moreover

$$\begin{aligned} \|p - f^s\| &= \int_0^{\infty} |p_1(x) - f_1^s(x)|dx + \int_0^{\infty} |p_2(x) - f_2^s(x)|dx \\ &= \int_0^{2s} |f_1^s(x)|dx + \int_0^{2s} |f_2^s(x)|dx \\ &= \int_0^{s} |f_1^s(0)|\left(1 - \frac{x}{s}\right)^2 dx \\ &\quad + \int_s^{2s} |u_1|(x-s)^2(x-2s)^2 dx \\ &\quad + \int_0^{s} |f_2^s(0)|\left(1 - \frac{x}{s}\right)^2 dx \\ &\quad + \int_s^{2s} |u_2|(x-s)^2(x-2s)^2 dx \\ &= \sum_{i=1}^{2} |f_i^s(0)|\frac{s}{3} + \sum_{i=1}^{2} |u_i|\frac{s^5}{30} \to 0, \quad \text{as} \quad s \to 0. \end{aligned} \tag{3-32}$$

(3-32) shows that $\mathbb{L} \subset \overline{D(A)}$. Therefore, $D(A)$ is dense in X.

From the first step, the second step and the Hille-Yosida theorem (see Theorem 1.68) we know that A generates a C_0-semigroup.

In the following we will verify that U and E are bounded linear operators. For any $p \in X$, from the definitions of U and E we estimate

$$\begin{aligned} Up(x) &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & \lambda\eta_1 & 0 \end{pmatrix} \begin{pmatrix} p_0 \\ p_1(x) \\ p_2(x) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \lambda\eta_1 p_1(x) \end{pmatrix} \\ &\Longrightarrow \\ \|Up\| &= \int_0^\infty |\lambda\eta_1 p_1(x)| dx = \lambda\eta_1 \int_0^\infty |p_1(x)| dx \\ &= \lambda\eta_1 \|p_1\|_{L^1[0,\infty)} \leq \lambda\eta_1 \|p\|. \end{aligned} \tag{3-33}$$

$$\begin{aligned} Ep(x) &= \begin{pmatrix} \eta_2 \int_0^\infty \mu(x) p_1(x) dx \\ 0 \\ 0 \end{pmatrix} \\ &\Longrightarrow \\ \|Ep\| &= \left| \eta_2 \int_0^\infty \mu(x) p_1(x) dx \right| \leq \eta_2 \int_0^\infty |\mu(x) p_1(x)| dx \\ &\leq M\eta_2 \int_0^\infty |p_1(x)| dx = M\eta_2 \|p_1\|_{L^1[0,\infty)} \\ &\leq M\|p\|. \end{aligned} \tag{3-34}$$

(3-33) and (3-34) show that U and E are bounded operators. It is not difficult to verify U and E are linear operators. So, U and E are bounded linear operators. This together with the first step, the second step and the perturbation theorem of a C_0-semigroup (see Theorem 1.80) imply that $A+U+E$ generates a C_0-semigroup $T(t)$.

Lastly we will prove that $A+U+E$ is a dispersive operator (see Definition 1.74). For any $p \in D(A)$ we take

$$\phi(x) = \left(\frac{[p_0]^+}{p_0}, \frac{[p_1(x)]^+}{p_1(x)}, \frac{[p_2(x)]^+}{p_2(x)} \right),$$

where

$$[p_0]^+ = \begin{cases} p_0 & \text{as} \quad p_0 > 0 \\ 0 & \text{as} \quad p_0 \leq 0 \end{cases}$$

$$[p_i(x)]^+ = \begin{cases} p_i(x) & \text{as} \quad p_i(x) > 0 \\ 0 & \text{as} \quad p_i(x) \leq 0 \end{cases}, i = 1, 2.$$

Let $V_i = \{x \in [0,\infty) \mid p_i(x) > 0\}$ and $W_i = \{x \in [0,\infty) \mid p_i(x) \le 0\}$ for $i = 1,2$, then we calculate

$$\begin{aligned}
\int_0^\infty \frac{dp_i(x)}{dx}\frac{[p_i(x)]^+}{p_i(x)}dx &= \int_{V_i} \frac{dp_i(x)}{dx}\frac{[p_i(x)]^+}{p_i(x)}dx \\
&\quad + \int_{W_i} \frac{dp_i(x)}{dx}\frac{[p_i(x)]^+}{p_i(x)}dx \\
&= \int_{V_i} \frac{dp_i(x)}{dx}\frac{[p_i(x)]^+}{p_i(x)}dx \\
&= \int_{V_i} \frac{dp_i(x)}{dx}dx = \int_0^\infty \frac{d[p_i(x)]^+}{dx}dx \\
&= [p_i(x)]^+\Big|_0^\infty \\
&= -[p_i(0)]^+, \quad i = 1,2.
\end{aligned} \tag{3-35}$$

$$\int_0^\infty p_1(x)\frac{[p_2(x)]^+}{p_2(x)}dx \le \int_0^\infty [p_1(x)]^+ dx. \tag{3-36}$$

For $p \in D(A)$ and $\phi(x)$, by using the boundary conditions on p, (3-35), (3-36) and $q_2 + \eta_2 = 1$ we have

$$\begin{aligned}
&\langle (A+U+E)p, \phi\rangle \\
&= \left\{-\lambda\eta_1 p_0 + \eta_2 \int_0^\infty \mu(x)p_1(x)dx\right\}\frac{[p_0]^+}{p_0} \\
&\quad - \int_0^\infty \left\{\frac{dp_1(x)}{dx} + (\lambda\eta_1 + \mu(x))p_1(x)\right\}\frac{[p_1(x)]^+}{p_1(x)}dx \\
&\quad + \int_0^\infty \left\{-\frac{dp_2(x)}{dx} - \mu(x)p_2(x) + \lambda\eta_1 p_1(x)\right\}\frac{[p_2(x)]^+}{p_2(x)}dx \\
&= -\lambda\eta_1[p_0]^+ + \eta_2\frac{[p_0]^+}{p_0}\int_0^\infty \mu(x)p_1(x)dx \\
&\quad - \int_0^\infty \frac{dp_1(x)}{dx}\frac{[p_1(x)]^+}{p_1(x)}dx - \int_0^\infty (\lambda\eta_1 + \mu(x))[p_1(x)]^+dx \\
&\quad - \int_0^\infty \frac{dp_2(x)}{dx}\frac{[p_2(x)]^+}{p_2(x)}dx - \int_0^\infty \mu(x)[p_2(x)]^+dx \\
&\quad + \lambda\eta_1\int_0^\infty p_1(x)\frac{[p_2(x)]^+}{p_2(x)}dx \\
&= -\lambda\eta_1[p_0]^+ + \eta_2\frac{[p_0]^+}{p_0}\int_0^\infty \mu(x)p_1(x)dx \\
&\quad + [p_1(0)]^+ - \int_0^\infty (\lambda\eta_1 + \mu(x))[p_1(x)]^+dx \\
&\quad + [p_2(0)]^+ - \int_0^\infty \mu(x)[p_2(x)]^+dx
\end{aligned}$$

$$\begin{aligned}
&+\lambda\eta_1\int_0^\infty p_1(x)\frac{[p_2(x)]^+}{p_2(x)}dx\\
&\le -\lambda\eta_1[p_0]^+ + \eta_2\frac{[p_0]^+}{p_0}\int_0^\infty \mu(x)p_1(x)dx\\
&+\lambda\eta_1[p_0]^+ + q_2\int_0^\infty \mu(x)[p_1(x)]^+dx\\
&+\eta_2\int_0^\infty \mu(x)[p_2(x)]^+dx - \int_0^\infty(\lambda\eta_1+\mu(x))[p_1(x)]^+dx\\
&+q_2\int_0^\infty \mu(x)[p_2(x)]^+dx - \int_0^\infty \mu(x)[p_2(x)]^+dx\\
&+\lambda\eta_1\int_0^\infty [p_1(x)]^+dx\\
&=\left[\eta_2\frac{[p_0]^+}{p_0}+q_2-1\right]\int_0^\infty \mu(x)[p_1(x)]^+dx\\
&+(\eta_2+q_2-1)\int_0^\infty \mu(x)[p_2(x)]^+dx\\
&=\left[\eta_2\frac{[p_0]^+}{p_0}-\eta_2\right]\int_0^\infty \mu(x)[p_1(x)]^+dx\\
&\le 0. \qquad\qquad (3\text{-}37)
\end{aligned}$$

(3-37) shows that $A+U+E$ is a dispersive operator by Definition 1.74. From which together with the first step, the second step and Theorem 1.79 we know that $A+U+E$ generates a positive contraction C_0-semigroup. By the uniqueness of a C_0-semigroup (see Remark 1.66) it follows that $T(t)$ is a positive contraction C_0-semigroup. □

Corollary 3.2. *If $M=\sup_{x\in[0,\infty)}\mu(x)<\infty$, then $A+U$ generates a positive contraction C_0-semigroup $S(t)$.*

Proof. For $p\in D(A)$ and $\phi(x)$ in Theorem 3.1 it is the same as (3-37)

$$\begin{aligned}
\langle (A+U)p,\phi\rangle &= -\lambda\eta_1 p_0\frac{[p_0]^+}{p_0}\\
&-\int_0^\infty\left\{\frac{dp_1(x)}{dx}+(\lambda\eta_1+\mu(x))p_1(x)\right\}\frac{[p_1(x)]^+}{p_1(x)}dx\\
&+\int_0^\infty\left\{-\frac{dp_2(x)}{dx}-\mu(x)p_2(x)+\lambda\eta_1p_1(x)\right\}\frac{[p_2(x)]^+}{p_2(x)}dx\\
&=-\lambda\eta_1[p_0]^+\\
&-\int_0^\infty\frac{dp_1(x)}{dx}\frac{[p_1(x)]^+}{p_1(x)}dx-\int_0^\infty(\lambda\eta_1+\mu(x))[p_1(x)]^+dx
\end{aligned}$$

$$
\begin{aligned}
&-\int_0^\infty \frac{dp_2(x)}{dx}\frac{[p_2(x)]^+}{p_2(x)}dx - \int_0^\infty \mu(x)[p_2(x)]^+dx \\
&+\lambda\eta_1\int_0^\infty p_1(x)\frac{[p_2(x)]^+}{p_2(x)}dx \\
=\;& -\lambda\eta_1[p_0]^+ \\
&+[p_1(0)]^+ - \int_0^\infty (\lambda\eta_1+\mu(x))[p_1(x)]^+dx \\
&+[p_2(0)]^+ - \int_0^\infty \mu(x)[p_2(x)]^+dx \\
&+\lambda\eta_1\int_0^\infty p_1(x)\frac{[p_2(x)]^+}{p_2(x)}dx \\
\le\;& -\lambda\eta_1[p_0]^+ \\
&+\lambda\eta_1[p_0]^+ + q_2\int_0^\infty \mu(x)[p_1(x)]^+dx \\
&+\eta_2\int_0^\infty \mu(x)[p_2(x)]^+dx - \int_0^\infty (\lambda\eta_1+\mu(x))[p_1(x)]^+dx \\
&+q_2\int_0^\infty \mu(x)[p_2(x)]^+dx - \int_0^\infty \mu(x)[p_2(x)]^+dx \\
&+\lambda\eta_1\int_0^\infty [p_1(x)]^+dx \\
=\;& (q_2-1)\int_0^\infty \mu(x)[p_1(x)]^+dx \\
&+(\eta_2+q_2-1)\int_0^\infty \mu(x)[p_2(x)]^+dx \\
=\;& (q_2-1)\int_0^\infty \mu(x)[p_1(x)]^+dx \\
\le\;& 0.
\end{aligned}
$$

From which together with the first step and the second step of Theorem 3.1 and Theorem 1.79 we know that $A+U$ generates a positive contraction C_0-semigroup $S(t)$. □

Theorem 3.3. *If* $M = \sup_{x\in[0,\infty)} \mu(x) < \infty$, *then the system* (3-15) *has a unique positive time-dependent solution* $p(x,t)$ *satisfying*

$$\|p(\cdot,t)\| \le 1, \quad \forall t\in[0,\infty). \tag{3-38}$$

Proof. Since the initial value $p(0) = (1,0,0) \in D(A)$, from Theorem 3.1 and Theorem 1.81 we know that the system (3-15) has a unique positive time-dependent solution $p(x,t)$ which can be expressed as

$$p(x,t) = T(t)p(0), \quad t\in[0,\infty). \tag{3-39}$$

Theorem 3.1 implies $\|T(t)\| \leq 1$ for $\forall t \in [0, \infty)$, from which together with (3-39) we derive

$$\|p(\cdot, t)\| = \|T(t)p(0)\| \leq \|T(t)\| \|p(0)\| \leq \|p(0)\| \tag{3-40}$$
$$= |1| + \int_0^\infty |0| dx + \int_0^\infty |0| dx = 1, \quad \forall t \in [0, \infty). \qquad \square$$

(3-38) or (3-40) reflects the physical background of $p(x, t)$.

Remark 3.4. One can verify that $T(t)$ is not isometric for the initial value of the system (3-15), that is, $\|T(t)p(0)\| \neq \|p(0)\|, \quad \forall t \in (0, \infty)$. This differs from queueing models (see Gupur et al. [65]).

3.3 Asymptotic Behavior of the Time-dependent Solution of the System (3-15)

In this section, we first prove $S(t)$ is a quasi-compact operator by studying two operators $V(t)$ and $W(t)$, then by using compactness of E we obtain $T(t)$ is a quasi-compact operator, next we prove that 0 is an eigenvalue of $A + U + E$ and $(A+U+E)^*$ with geometric multiplicity 1, thus by using Theorem 1.90 we deduce that $T(t)$ exponentially converges to a project operator $\mathbb{P}$ and the time-dependent solution of the system (3-15) strongly converges to its steady-state solution. Lastly, when $\mu(x)$ is a constant, by determining the expression of $\mathbb{P}$ we obtain that the time-dependent solution of the system (3-15) exponentially converges to its steady-state solution.

Proposition 3.5. *If for $\varphi \in X$, $p(x, t) = (S(t)\varphi)(x)$ is a solution of the following system (see Corollary 3.2)*

$$\begin{cases} \frac{dp(t)}{dt} = (A + U)p(t), \quad t > 0, \\ p(0) = \varphi, \end{cases} \tag{3-41}$$

then $p(x, t) = (S(t)\varphi)(x)$

$$= \begin{pmatrix} \varphi_0 e^{-\lambda\eta_1 t} \\ p_1(0, t - x)e^{-\lambda\eta_1 x - \int_0^x \mu(\tau)d\tau} \\ p_2(0, t - x)e^{-\int_0^x \mu(\tau)d\tau} + p_1(0, t - x)e^{-\int_0^x \mu(\tau)d\tau}\left(1 - e^{-\lambda\eta_1 x}\right) \end{pmatrix}$$

when $x < t$; $p(x, t) = (S(t)\varphi)(x)$

$$= \begin{pmatrix} \varphi_0 e^{-\lambda\eta_1 t} \\ \varphi_1(x - t)e^{-\lambda\eta_1 t - \int_{x-t}^x \mu(\tau)d\tau} \\ \varphi_2(x - t)e^{-\int_{x-t}^x \mu(\tau)d\tau} + \varphi_1(x - t)e^{-\int_{x-t}^x \mu(\tau)d\tau}\left(1 - e^{-\lambda\eta_1 t}\right) \end{pmatrix}$$

when $x > t$, where $p_1(0, t - x)$ and $p_2(0, t - x)$ are given by (3-10) *and* (3-11).

Proof. p is a solution of the following system:

$$\frac{dp_0(t)}{dt} = -\lambda\eta_1 p_0(t), \tag{3-42}$$
$$\frac{\partial p_1(x,t)}{\partial t} + \frac{\partial p_1(x,t)}{\partial x} = -(\lambda\eta_1 + \mu(x))p_1(x,t), \tag{3-43}$$
$$\frac{\partial p_2(x,t)}{\partial t} + \frac{\partial p_2(x,t)}{\partial x} = -\mu(x)p_2(x,t) + \lambda\eta_1 p_1(x,t), \tag{3-44}$$
$$\begin{aligned} p_1(0,t) = \lambda\eta_1 p_0(t) + q_2\int_0^\infty \mu(x)p_1(x,t)dx \\ + \eta_2\int_0^\infty \mu(x)p_2(x,t)dx, \end{aligned} \tag{3-45}$$
$$p_2(0,t) = q_2\int_0^\infty \mu(x)p_2(x,t)dx, \tag{3-46}$$
$$p_0(0) = \varphi_0, \quad p_i(x,0) = \varphi_i(x), \quad i = 1,2. \tag{3-47}$$

If we set $\xi = x - t$ and define $Q_1(t) = p_1(\xi + t, t)$ and $Q_2(t) = p_2(\xi + t, t)$, then (3-43) and (3-44) are equivalent to

$$\frac{dQ_1(t)}{dt} = -(\lambda\eta_1 + \mu(\xi + t))Q_1(t), \tag{3-48}$$
$$\frac{dQ_2(t)}{dt} = -\mu(\xi + t)Q_2(t) + \lambda\eta_1 Q_1(t). \tag{3-49}$$

If $\xi < 0$ (equivalently $x < t$), then integrating (3-48)–(3-49) from $-\xi$ to t and using $Q_1(-\xi) = p_1(0,-\xi) = p_1(0,t-x)$, $Q_2(-\xi) = p_2(0,-\xi) = p_2(0,t-x)$ we have

$$\begin{aligned} p_1(x,t) &= Q_1(t) = Q_1(-\xi)e^{-\lambda\eta_1(t+\xi) - \int_{-\xi}^t \mu(\xi+\tau)d\tau} \\ &\overset{\xi+\tau=y}{=\!=\!=} p_1(0,t-x)e^{-\lambda\eta_1 x - \int_0^{t+\xi}\mu(y)dy} \\ &= p_1(0,t-x)e^{-\lambda\eta_1 x - \int_0^x \mu(\tau)d\tau}. \end{aligned} \tag{3-50}$$

$$\begin{aligned} p_2(x,t) &= Q_2(t) = Q_2(-\xi)e^{-\int_{-\xi}^t \mu(\xi+\tau)d\tau} \\ &\quad + \lambda\eta_1 e^{-\int_{-\xi}^t \mu(\xi+\tau)d\tau}\int_{-\xi}^t Q_1(\eta)e^{\int_{-\xi}^\eta \mu(\xi+\tau)d\tau}d\eta \\ &\overset{\xi+\tau=y}{=\!=\!=} p_2(0,t-x)e^{-\int_0^{t+\xi}\mu(y)dy} \\ &\quad + \lambda\eta_1 e^{-\int_0^{t+\xi}\mu(y)dy}\int_{-\xi}^t Q_1(\eta)e^{\int_0^{\eta+\xi}\mu(y)dy}d\eta \\ &\overset{\eta+\xi=v}{=\!=\!=} p_2(0,t-x)e^{-\int_0^x \mu(y)dy} \\ &\quad + \lambda\eta_1 e^{-\int_0^x \mu(y)dy}\int_0^{t+\xi} Q_1(v-\xi)e^{\int_0^v \mu(y)dy}dv \end{aligned}$$

$$
\begin{aligned}
&\xlongequal{Q_1(t)=p_1(\xi+t,t)} p_2(0,t-x)e^{-\int_0^x \mu(\tau)d\tau} \\
&\quad + \lambda\eta_1 e^{-\int_0^x \mu(\tau)d\tau} \int_0^x p_1(\xi+v-\xi, v-\xi)e^{\int_0^v \mu(\tau)d\tau} dv \\
&= p_2(0,t-x)e^{-\int_0^x \mu(\tau)d\tau} \\
&\quad + \lambda\eta_1 e^{-\int_0^x \mu(\tau)d\tau} \int_0^x p_1(y, y-\xi)e^{\int_0^y \mu(\tau)d\tau} dy.
\end{aligned} \tag{3-51}
$$

Since $\xi < 0$, $y - \xi > y$ and therefore using (3-50) we deduce

$$
\begin{aligned}
p_1(y, y-\xi) &= p_1(0, y-\xi-y)e^{-\lambda\eta_1 y - \int_0^y \mu(\tau)d\tau} \\
&= p_1(0, -\xi)e^{-\lambda\eta_1 y - \int_0^y \mu(\tau)d\tau}.
\end{aligned}
$$

From which together with (3-51) we have

$$
\begin{aligned}
p_2(x,t) &= p_2(0,t-x)e^{-\int_0^x \mu(\tau)d\tau} \\
&\quad + \lambda\eta_1 e^{-\int_0^x \mu(\tau)d\tau} \int_0^x p_1(0,-\xi)e^{-\lambda\eta_1 y} dy \\
&= p_2(0,t-x)e^{-\int_0^x \mu(\tau)d\tau} \\
&\quad + p_1(0,t-x)e^{-\int_0^x \mu(\tau)d\tau}\left(1 - e^{-\lambda\eta_1 x}\right).
\end{aligned} \tag{3-52}
$$

From (3-42) and (3-47) we obtain

$$
p_0(t) = \varphi_0 e^{-\lambda\eta_1 t}. \tag{3-53}
$$

If $\xi > 0$ (equivalently $x > t$), then integrating (3-48) and (3-49) from 0 to t and then using relations $Q_1(0) = p_1(\xi, 0) = \varphi_1(x-t)$ and $Q_2(0) = p_2(\xi, 0) = \varphi_2(x-t)$, and making a similar argument to (3-50), (3-51) and (3-52) we derive

$$
\begin{aligned}
p_1(x,t) &= Q_1(t) = Q_1(0)e^{-\lambda\eta_1 t - \int_0^t \mu(\xi+\tau)d\tau} \\
&\xlongequal{\xi+\tau=\eta} \varphi_1(x-t)e^{-\lambda\eta_1 t - \int_\xi^{\xi+t} \mu(\eta)d\eta} \\
&= \varphi_1(x-t)e^{-\lambda\eta_1 t - \int_{x-t}^x \mu(\tau)d\tau}.
\end{aligned} \tag{3-54}
$$

$$
\begin{aligned}
p_2(x,t) &= Q_2(t) = Q_2(0)e^{-\int_0^t \mu(\xi+\tau)d\tau} \\
&\quad + \lambda\eta_1 e^{-\int_0^t \mu(\xi+\tau)d\tau} \int_0^t Q_1(z)e^{\int_0^z \mu(\xi+\tau)d\tau} dz \\
&\xlongequal{Q_1(z)=Q_1(0)e^{-\lambda\eta_1 z - \int_0^z \mu(\xi+\tau)d\tau}} Q_2(0)e^{-\int_0^t \mu(\xi+\tau)d\tau} \\
&\quad + \lambda\eta_1 e^{-\int_0^t \mu(\xi+\tau)d\tau} \\
&\quad \times \int_0^t Q_1(0)e^{-\lambda\eta_1 z - \int_0^z \mu(\xi+\tau)d\tau} e^{\int_0^z \mu(\xi+\tau)d\tau} dz
\end{aligned}
$$

$$
\begin{aligned}
&= Q_2(0)e^{-\int_0^t \mu(\xi+\tau)d\tau} \\
&\quad + \lambda\eta_1 e^{-\int_0^t \mu(\xi+\tau)d\tau}\int_0^t Q_1(0)e^{-\lambda\eta_1 z}dz \\
&\xlongequal{\xi+\tau=y} Q_2(0)e^{-\int_\xi^{t+\xi}\mu(y)dy} \\
&\quad + Q_1(0)e^{-\int_\xi^{t+\xi}\mu(y)dy}\left(1-e^{-\lambda\eta_1 t}\right) \\
&= \phi_2(x-t)e^{-\int_{x-t}^{x}\mu(\tau)d\tau} \\
&\quad + \phi_1(x-t)\left(1-e^{-\lambda\eta_1 t}\right)e^{-\int_{x-t}^{x}\mu(\tau)d\tau}.
\end{aligned}
\tag{3-55}
$$

(3-50) and (3-52)–(3-55) show that the result of this proposition is right. □

If we define two operators as follows, for $p \in X$,

$$
(V(t)p)(x) = \begin{cases} 0 & x \in [0,t), \\ (S(t)p)(x) & x \in [t,\infty), \end{cases} \tag{3-56}
$$

$$
(W(t)p)(x) = \begin{cases} (S(t)p)(x) & x \in [0,t), \\ 0 & x \in [t,\infty), \end{cases} \tag{3-57}
$$

then $S(t)p = V(t)p + W(t)p,\ \forall p \in X$.

Theorem 3.6. *Assume that $\mu(x)$ is Lipschitz continuous and there exist two positive constants $\overline{\mu}$ and $\underline{\mu}$ such that $0 < \underline{\mu} \le \mu(x) \le \overline{\mu} < \infty$; then $W(t)$ is a compact operator on X.*

Proof. According to the definition of $W(t)$, it suffices to prove the condition (1) in Corollary 1.37. For bounded $\varphi \in X$ we set $p(x,t) = (S(t)\varphi)(x),\ x \in [0,t)$, then by Corollary 3.2 and Theorem 1.81 $p(x,t)$ is a solution of the system (3-41). So, by using the result of Proposition 3.5 we have, for $x, h \in [0,t),\ x+h \in [0,t)$,

$$
\begin{aligned}
&\int_0^t |p_1(x+h,t)-p_1(x,t)|dx \\
&\quad + \int_0^t |p_2(x+h,t)-p_2(x,t)|dx \\
&= \int_0^t \Big| p_1(0,t-x-h)e^{-\lambda\eta_1(x+h)-\int_0^{x+h}\mu(\tau)d\tau} \\
&\quad - p_1(0,t-x)e^{-\lambda\eta_1 x-\int_0^{x}\mu(\tau)d\tau}\Big| dx \\
&\quad + \int_0^t \Big| p_2(0,t-x-h)e^{-\int_0^{x+h}\mu(\tau)d\tau} \\
&\quad + p_1(0,t-x-h)e^{-\int_0^{x+h}\mu(\tau)d\tau}\left(1-e^{-\lambda\eta_1(x+h)}\right) \\
&\quad - p_2(0,t-x)e^{-\int_0^{x}\mu(\tau)d\tau}
\end{aligned}
$$

$$
\begin{aligned}
&-p_1(0,t-x)e^{-\int_0^x \mu(\tau)d\tau}\left(1-e^{-\lambda\eta_1 x}\right)\Big| dx\\
&= \int_0^t \Big| p_1(0,t-x-h)e^{-\lambda\eta_1(x+h)-\int_0^{x+h}\mu(\tau)d\tau}\\
&\quad - p_1(0,t-x-h)e^{-\lambda\eta_1 x-\int_0^x \mu(\tau)d\tau}\\
&\quad + p_1(0,t-x-h)e^{-\lambda\eta_1 x-\int_0^x \mu(\tau)d\tau}\\
&\quad -p_1(0,t-x)e^{-\lambda\eta_1 x-\int_0^x \mu(\tau)d\tau}\Big| dx\\
&\quad + \int_0^t \Big| p_2(0,t-x-h)e^{-\int_0^{x+h}\mu(\tau)d\tau}\\
&\quad - p_2(0,t-x-h)e^{-\int_0^x \mu(\tau)d\tau}\\
&\quad + p_2(0,t-x-h)e^{-\int_0^x \mu(\tau)d\tau} - p_2(0,t-x)e^{-\int_0^x \mu(\tau)d\tau}\\
&\quad +p_1(0,t-x-h)e^{-\int_0^{x+h}\mu(\tau)d\tau}\left(1-e^{-\lambda\eta_1(x+h)}\right)\\
&\quad - p_1(0,t-x-h)e^{-\int_0^x \mu(\tau)d\tau}\left(1-e^{-\lambda\eta_1 x}\right)\\
&\quad + p_1(0,t-x-h)e^{-\int_0^x \mu(\tau)d\tau}\left(1-e^{-\lambda\eta_1 x}\right)\\
&\quad -p_1(0,t-x)e^{-\int_0^x \mu(\tau)d\tau}\left(1-e^{-\lambda\eta_1 x}\right)\Big| dx\\
&\le \int_0^t |p_1(0,t-x-h)| \Big| e^{-\lambda\eta_1(x+h)-\int_0^{x+h}\mu(\tau)d\tau}\\
&\quad -e^{-\lambda\eta_1 x-\int_0^x \mu(\tau)d\tau}\Big| dx\\
&\quad + \int_0^t |p_1(0,t-x-h)-p_1(0,t-x)| e^{-\lambda\eta_1 x-\int_0^x \mu(\tau)d\tau}dx\\
&\quad + \int_0^t |p_2(0,t-x-h)| \Big| e^{-\int_0^{x+h}\mu(\tau)d\tau} - e^{-\int_0^x \mu(\tau)d\tau}\Big| dx\\
&\quad + \int_0^t |p_2(0,t-x-h)-p_2(0,t-x)| e^{-\int_0^x \mu(\tau)d\tau}dx\\
&\quad + \int_0^t |p_1(0,t-x-h)| \Big| e^{-\int_0^{x+h}\mu(\tau)d\tau}\left(1-e^{-\lambda\eta_1(x+h)}\right)\\
&\quad -e^{-\int_0^x \mu(\tau)d\tau}\left(1-e^{-\lambda\eta_1 x}\right)\Big| dx\\
&\quad + \int_0^t |p_1(0,t-x-h)-p_1(0,t-x)|\\
&\quad \times e^{-\int_0^x \mu(\tau)d\tau}\left(1-e^{-\lambda\eta_1 x}\right)dx.
\end{aligned}
\tag{3-58}
$$

If we can verify that the right side of (3-58) uniformly converges to zero, then we deduce our desired result. To do this, in the following we will estimate each term in (3-58). By using (3-45), Corollary 3.2 and Theorem 1.81 we have

$$\begin{aligned}
&|p_1(0,t-x-h)| \\
&= \bigg|\lambda\eta_1 p_0(t-x-h) + q_2\int_0^\infty \mu(s)p_1(s,t-x-h)ds \\
&\quad +\eta_2\int_0^\infty \mu(s)p_2(s,t-x-h)ds\bigg| \\
&\le |\lambda\eta_1 p_0(t-x-h)| + \overline{\mu}q_2\int_0^\infty |p_1(s,t-x-h)|ds \\
&\quad +\overline{\mu}\eta_2\int_0^\infty |p_2(s,t-x-h)|ds \\
&\le \max\{\lambda\eta_1,\overline{\mu}q_2,\overline{\mu}\eta_2\}\Big\{|p_0(t-x-h)| \\
&\quad +\int_0^\infty |p_1(s,t-x-h)|ds + \int_0^\infty |p_2(s,t-x-h)|ds\Big\} \\
&= \max\{\lambda\eta_1,\overline{\mu}q_2,\overline{\mu}\eta_2\}\|p(\cdot,t-x-h)\|_X \\
&= \max\{\lambda\eta_1,\overline{\mu}q_2,\overline{\mu}\eta_2\}\|S(t-x-h)\varphi(\cdot)\|_X \\
&\le \max\{\lambda\eta_1,\overline{\mu}q_2,\overline{\mu}\eta_2\}\|\varphi\|.
\end{aligned} \tag{3-59}$$

By (3-59) we estimate the first term and fifth term of (3-58) as follows.

$$\begin{aligned}
&\int_0^t |p_1(0,t-x-h)|\left|e^{-\lambda\eta_1(x+h)-\int_0^{x+h}\mu(\tau)d\tau} - e^{-\lambda\eta_1 x-\int_0^x \mu(\tau)d\tau}\right|dx \\
&\le \max\{\lambda\eta_1,\overline{\mu}q_2,\overline{\mu}\eta_2\}\|\varphi\| \\
&\quad\times\int_0^t \left|e^{-\lambda\eta_1(x+h)-\int_0^{x+h}\mu(\tau)d\tau} - e^{-\lambda\eta_1 x-\int_0^x \mu(\tau)d\tau}\right|dx \\
&\to 0 \quad \text{as} \quad |h|\to 0, \quad \text{uniformly for } \varphi.
\end{aligned} \tag{3-60}$$

$$\begin{aligned}
&\int_0^t |p_1(0,t-x-h)|\bigg|e^{-\int_0^{x+h}\mu(\tau)d\tau}\left(1-e^{-\lambda\eta_1(x+h)}\right) \\
&\quad -e^{-\int_0^x \mu(\tau)d\tau}\left(1-e^{-\lambda\eta_1 x}\right)\bigg|dx \\
&\le \max\{\lambda\eta_1,\overline{\mu}q_2,\overline{\mu}\eta_2\}\|\varphi\|\int_0^t \bigg|e^{-\int_0^{x+h}\mu(\tau)d\tau}\left(1-e^{-\lambda\eta_1(x+h)}\right) \\
&\quad -e^{-\int_0^x \mu(\tau)d\tau}\left(1-e^{-\lambda\eta_1 x}\right)\bigg|dx \\
&\to 0 \quad \text{as} \quad |h|\to 0, \quad \text{uniformly for} \quad \varphi.
\end{aligned} \tag{3-61}$$

By using (3-46), Corollary 3.2 and Theorem 1.81 it gives

$$|p_2(0,t-x-h)| = \left|q_2\int_0^\infty \mu(s)p_2(s,t-x-h)ds\right|$$

$$
\begin{aligned}
&\le \overline{\mu} q_2 \int_0^\infty |p_2(s, t-x-h)| ds \\
&= \overline{\mu} q_2 \|p_2(\cdot, t-x-h)\|_{L^1[0,\infty)} \\
&\le \overline{\mu} q_2 \|p(\cdot, t-x-h)\|_X \\
&= \overline{\mu} q_2 \|S(t-x-h)\varphi(\cdot)\|_X \\
&\le \overline{\mu} q_2 \|\varphi\|.
\end{aligned} \tag{3-62}
$$

By using (3-62) we estimate the third term in (3-58) as follows.

$$
\begin{aligned}
&\int_0^t |p_2(0, t-x-h)| \left| e^{-\int_0^{x+h}\mu(\tau)d\tau} - e^{-\int_0^x \mu(\tau)d\tau} \right| dx \\
&\le \overline{\mu} q_2 \|\varphi\| \int_0^t \left| e^{-\int_0^{x+h}\mu(\tau)d\tau} - e^{-\int_0^x \mu(\tau)d\tau} \right| dx \\
&\to 0 \quad \text{as} \quad |h| \to 0, \quad \text{uniformly for} \quad \varphi.
\end{aligned} \tag{3-63}
$$

By applying (3-45), Proposition 3.5 and the following formulas

$$
\begin{aligned}
&\int_0^{t-x-h} \mu(s) p_i(s, t-x-h) ds \xlongequal{t-x-h-s=y} \\
&\int_{t-x-h}^0 -\mu(t-x-h-y) p_i(t-x-h-y, t-x-h) dy \\
&= \int_0^{t-x-h} \mu(t-x-h-y) p_i(t-x-h-y, t-x-h) dy, \\
&\quad i = 1, 2,
\end{aligned}
$$

$$
\begin{aligned}
&\int_0^{t-x} \mu(s) p_i(s, t-x) ds \\
&\xlongequal{t-x-s=y} \int_{t-x}^0 -\mu(t-x-y) p_i(t-x-y, t-x) dy \\
&= \int_0^{t-x} \mu(t-x-y) p_i(t-x-y, t-x) dy, \quad i = 1, 2
\end{aligned}
$$

we have

$$
\begin{aligned}
&|p_1(0, t-x-h) - p_1(0, t-x)| \\
&= \Big| \lambda \eta_1 p_0(t-x-h) + q_2 \int_0^\infty \mu(s) p_1(s, t-x-h) ds \\
&\quad + \eta_2 \int_0^\infty \mu(s) p_2(s, t-x-h) ds - \lambda \eta_1 p_0(t-x) \\
&\quad - q_2 \int_0^\infty \mu(s) p_1(s, t-x) ds - \eta_2 \int_0^\infty \mu(s) p_2(s, t-x) ds \Big|
\end{aligned}
$$

$$\begin{aligned}
&\leq \lambda\eta_1|p_0(t-x-h)-p_0(t-x)| \\
&\quad + q_2\left|\int_0^{\infty}\mu(s)p_1(s,t-x-h)ds\right. \\
&\quad \left. -\int_0^{\infty}\mu(s)p_1(s,t-x)ds\right| \\
&\quad + \eta_2\left|\int_0^{\infty}\mu(s)p_2(s,t-x-h)ds\right. \\
&\quad \left. -\int_0^{\infty}\mu(s)p_2(s,t-x)ds\right| \\
&= \lambda\eta_1\left|\varphi_0 e^{-\lambda\eta_1(t-x-h)}-\varphi_0 e^{-\lambda\eta_1(t-x)}\right| \\
&\quad + q_2\left|\int_0^{t-x-h}\mu(s)p_1(s,t-x-h)ds\right. \\
&\quad + \int_{t-x-h}^{\infty}\mu(s)p_1(s,t-x-h)ds \\
&\quad \left. -\int_0^{t-x}\mu(s)p_1(s,t-x)ds-\int_{t-x}^{\infty}\mu(s)p_1(s,t-x)ds\right| \\
&\quad + \eta_2\left|\int_0^{t-x-h}\mu(s)p_2(s,t-x-h)ds\right. \\
&\quad + \int_{t-x-h}^{\infty}\mu(s)p_2(s,t-x-h)ds \\
&\quad \left. -\int_0^{t-x}\mu(s)p_2(s,t-x)ds-\int_{t-x}^{\infty}\mu(s)p_2(s,t-x)ds\right| \\
&= \lambda\eta_1|\varphi_0|\left|e^{-\lambda\eta_1(t-x-h)}-e^{-\lambda\eta_1(t-x)}\right| \\
&+ q_2\left|\int_0^{t-x-h}\mu(t-x-h-y)p_1(t-x-h-y,t-x-h)dy\right. \\
&+ \int_{t-x-h}^{\infty}\mu(s)\varphi_1(s-t+x+h)e^{-\lambda\eta_1(t-x-h)-\int_{s-t+x+h}^{s}\mu(\tau)d\tau}ds \\
&- \int_0^{t-x}\mu(t-x-y)p_1(t-x-y,t-x)dy \\
&\left. - \int_{t-x}^{\infty}\mu(s)\varphi_1(s-t+x)e^{-\lambda\eta_1(t-x)-\int_{s-t+x}^{s}\mu(\tau)d\tau}ds\right| \\
&+ \eta_2\left|\int_0^{t-x-h}\mu(t-x-h-y)p_2(t-x-h-y,t-x-h)dy\right.
\end{aligned}$$

$$
\begin{aligned}
&+\int_{t-x-h}^{\infty}\mu(s)\Big[\varphi_2(s-t+x+h)e^{-\int_{s-t+x+h}^{s}\mu(\tau)d\tau}\\
&+\varphi_1(s-t+x+h)e^{-\int_{s-t+x+h}^{s}\mu(\tau)d\tau}\left(1-e^{-\lambda\eta_1(t-x-h)}\right)\Big]\,ds\\
&-\int_0^{t-x}\mu(t-x-y)p_2(t-x-y,t-x)dy\\
&-\int_{t-x}^{\infty}\mu(s)\Big[\varphi_2(s-t+x)e^{-\int_{s-t+x}^{s}\mu(\tau)d\tau}\\
&+\ \varphi_1(s-t+x)e^{-\int_{s-t+x}^{s}\mu(\tau)d\tau}\left(1-e^{-\lambda\eta_1(t-x)}\right)\Big]\,ds\Big|\\
&\le\lambda\eta_1|\varphi_0|\left|e^{-\lambda\eta_1(t-x-h)}-e^{-\lambda\eta_1(t-x)}\right|\\
&+q_2\left|\int_0^{t-x-h}\mu(t-x-h-y)p_1(t-x-h-y,t-x-h)dy\right.\\
&\left.-\int_0^{t-x}\mu(t-x-y)p_1(t-x-y,t-x)dy\right|\\
&+q_2\left|\int_{t-x-h}^{\infty}\mu(s)\varphi_1(s-t+x+h)e^{-\lambda\eta_1(t-x-h)-\int_{s-t+x+h}^{s}\mu(\tau)d\tau}ds\right.\\
&\left.-\int_{t-x}^{\infty}\mu(s)\varphi_1(s-t+x)e^{-\lambda\eta_1(t-x)-\int_{s-t+x}^{s}\mu(\tau)d\tau}ds\right|\\
&+\eta_2\left|\int_0^{t-x-h}\mu(t-x-h-y)p_2(t-x-h-y,t-x-h)dy\right.\\
&\left.-\int_0^{t-x}\mu(t-x-y)p_2(t-x-y,t-x)dy\right|\\
&+\eta_2\left|\int_{t-x-h}^{\infty}\mu(s)\varphi_2(s-t+x+h)e^{-\int_{s-t+x+h}^{s}\mu(\tau)d\tau}ds\right.\\
&\left.-\int_{t-x}^{\infty}\mu(s)\varphi_2(s-t+x)e^{-\int_{s-t+x}^{s}\mu(\tau)d\tau}ds\right|\\
&+\eta_2\left|\int_{t-x-h}^{\infty}\mu(s)\varphi_1(s-t+x+h)e^{-\int_{s-t+x+h}^{s}\mu(\tau)d\tau}\right.\\
&\quad\times\left(1-e^{-\lambda\eta_1(t-x-h)}\right)ds\\
&-\int_{t-x}^{\infty}\mu(s)\varphi_1(s-t+x)e^{-\int_{s-t+x}^{s}\mu(\tau)d\tau}\\
&\quad\left.\times\left(1-e^{-\lambda\eta_1(t-x)}\right)ds\right|.
\end{aligned}
\tag{3-64}
$$

In the following we will estimate each term in (3-64). The third term in (3-64) satisfies, noting $\mu(x)$ is Lipschitz continuous (without loss of generality assume the Lipschitz constant is equal to 1)

$$
\begin{aligned}
&\bigg|\int_{t-x-h}^{\infty}\mu(s)\varphi_1(s-t+x+h)e^{-\lambda\eta_1(t-x-h)-\int_{s-t+x+h}^{s}\mu(\tau)d\tau}ds\\
&\quad-\int_{t-x}^{\infty}\mu(s)\varphi_1(s-t+x)e^{-\lambda\eta_1(t-x)-\int_{s-t+x}^{s}\mu(\tau)d\tau}ds\bigg|\\
&\underline{\underline{s-t+x+h=y,\ s-t+x=y}}\\
&\bigg|\int_0^{\infty}\mu(y+t-x-h)\varphi_1(y)e^{-\lambda\eta_1(t-x-h)-\int_y^{y+t-x-h}\mu(\tau)d\tau}dy\\
&\quad-\int_0^{\infty}\mu(y+t-x)\varphi_1(y)e^{-\lambda\eta_1(t-x)-\int_y^{y+t-x}\mu(\tau)d\tau}dy\bigg|\\
&=\bigg|\int_0^{\infty}\varphi_1(y)\Big[\mu(y+t-x-h)e^{-\lambda\eta_1(t-x-h)-\int_y^{y+t-x-h}\mu(\tau)d\tau}\\
&\quad-\mu(y+t-x)e^{-\lambda\eta_1(t-x)-\int_y^{y+t-x}\mu(\tau)d\tau}\Big]dy\bigg|\\
&=\bigg|\int_0^{\infty}\varphi_1(y)\Big[\mu(y+t-x-h)e^{-\lambda\eta_1(t-x-h)-\int_y^{y+t-x-h}\mu(\tau)d\tau}\\
&\quad-\mu(y+t-x-h)e^{-\lambda\eta_1(t-x)-\int_y^{y+t-x}\mu(\tau)d\tau}\\
&\quad+\mu(y+t-x-h)e^{-\lambda\eta_1(t-x)-\int_y^{y+t-x}\mu(\tau)d\tau}\\
&\quad-\mu(y+t-x)\ e^{-\lambda\eta_1(t-x)-\int_y^{y+t-x}\mu(\tau)d\tau}\Big]dy\bigg|\\
&=\bigg|\int_0^{\infty}\varphi_1(y)\Big[\mu(y+t-x-h)\Big(e^{-\lambda\eta_1(t-x-h)-\int_y^{y+t-x-h}\mu(\tau)d\tau}\\
&\quad-e^{-\lambda\eta_1(t-x)-\int_y^{y+t-x}\mu(\tau)d\tau}\Big)\\
&\quad+(\mu(y+t-x-h)-\mu(y+t-x))\\
&\qquad\times\ e^{-\lambda\eta_1(t-x)-\int_y^{y+t-x}\mu(\tau)d\tau}\Big]dy\bigg|\\
&\le\bigg\{\overline{\mu}\sup_{y\in[0,\infty)}\Big|e^{-\lambda\eta_1(t-x-h)-\int_y^{y+t-x-h}\mu(\tau)d\tau}\\
&\quad-e^{-\lambda\eta_1(t-x)-\int_y^{y+t-x}\mu(\tau)d\tau}\Big|\\
&\quad+|h|\sup_{y\in[0,\infty)}e^{-\lambda\eta_1(t-x)-\int_y^{y+t-x}\mu(\tau)d\tau}\bigg\}\|\varphi_1\|_{L^1[0,\infty)}\\
&\to 0\quad\text{as}\quad |h|\to 0,\quad\text{uniformly for}\quad\varphi.
\end{aligned}\tag{3-65}
$$

It is the same as the proof of (3-65); we estimate the sixth term of (3-64) as follows.

$$
\bigg|\int_{t-x-h}^{\infty}\mu(s)\varphi_1(s-t+x+h)e^{-\int_{s-t+x+h}^{s}\mu(\tau)d\tau}
$$

$$\times \left(1 - e^{-\lambda\eta_1(t-x-h)}\right) ds$$
$$- \int_{t-x}^{\infty} \mu(s)\varphi_1(s-t+x)e^{-\int_{s-t+x}^{s}\mu(\tau)d\tau} \left(1 - e^{-\lambda\eta_1(t-x)}\right) ds\Bigg|$$
$$\underline{\underline{s-t+x+h=y,\ s-t+x=y}}$$
$$\left|\int_0^{\infty} \mu(y+t-x-h)\varphi_1(y)e^{-\int_y^{y+t-x-h}\mu(\tau)d\tau} \left(1 - e^{-\lambda\eta_1(t-x-h)}\right) dy\right.$$
$$\left. - \int_0^{\infty} \mu(y+t-x)\varphi_1(y)e^{-\int_y^{y+t-x}\mu(\tau)d\tau} \left(1 - e^{-\lambda\eta_1(t-x)}\right) dy\right|$$
$$\le \int_0^{\infty} \mu(y+t-x-h)\left|e^{-\int_y^{y+t-x-h}\mu(\tau)d\tau} \left(1 - e^{-\lambda\eta_1(t-x-h)}\right)\right.$$
$$\left. - e^{-\int_y^{y+t-x}\mu(\tau)d\tau} \left(1 - e^{-\lambda\eta_1(t-x)}\right)\right| |\varphi_1(y)|dy$$
$$+ \int_0^{\infty} |\varphi_1(y)||\mu(y+t-x-h) - \mu(y+t-x)|$$
$$\times e^{-\int_y^{y+t-x}\mu(\tau)d\tau} \left(1 - e^{-\lambda\eta_1(t-x)}\right) dy$$
$$\le \left\{\overline{\mu} \sup_{y\in[0,\infty)} \left|e^{-\int_y^{y+t-x-h}\mu(\tau)d\tau} \left(1 - e^{-\lambda\eta_1(t-x-h)}\right)\right.\right.$$
$$\left. -e^{-\int_y^{y+t-x}\mu(\tau)d\tau} \left(1 - e^{-\lambda\eta_1(t-x)}\right)\right|$$
$$\left. + |h| \sup_{y\in[0,\infty)} e^{-\int_y^{y+t-x}\mu(\tau)d\tau} \left(1 - e^{-\lambda\eta_1(t-x)}\right)\right\} \|\varphi_1\|_{L^1[0,\infty)}.$$
$$\to 0 \quad as \quad |h| \to 0, \quad \text{uniformly for} \quad \varphi. \tag{3-66}$$

The fifth term in (3-64) satisfies, noting Lipschitz continuity of $\mu(x)$,

$$\left|\int_{t-x-h}^{\infty} \mu(s)\varphi_2(s-t+x+h)e^{-\int_{s-t+x+h}^{s}\mu(\tau)d\tau}ds\right.$$
$$\left. - \int_{t-x}^{\infty} \mu(s)\varphi_2(s-t+x)e^{-\int_{s-t+x}^{s}\mu(\tau)d\tau}ds\right|$$
$$\underline{\underline{s-t+x+h=y,\ s-t+x=y}}$$
$$\left|\int_0^{\infty} \mu(y+t-x-h)\varphi_2(y)e^{-\int_y^{y+t-x-h}\mu(\tau)d\tau}dy\right.$$
$$\left. - \int_0^{\infty} \mu(y+t-x)\varphi_2(y)e^{-\int_y^{y+t-x}\mu(\tau)d\tau}dy\right|$$
$$= \left|\int_0^{\infty} \varphi_2(y) \left[\mu(y+t-x-h)e^{-\int_y^{y+t-x-h}\mu(\tau)d\tau}\right.\right.$$
$$- \mu(y+t-x-h)e^{-\int_y^{y+t-x}\mu(\tau)d\tau}$$

$$
\begin{aligned}
&+\mu(y+t-x-h)e^{-\int_y^{y+t-x}\mu(\tau)d\tau}\\
&-\ \mu(y+t-x)e^{-\int_y^{y+t-x}\mu(\tau)d\tau}\Big]\,dy\Big|\\
=&\left|\int_0^\infty \varphi_2(y)\Big[\mu(y+t-x-h)\right.\\
&\times\left(e^{-\int_y^{y+t-x-h}\mu(\tau)d\tau}-e^{-\int_y^{y+t-x}\mu(\tau)d\tau}\right)\\
&+\ \left.(\mu(y+t-x-h)-\mu(y+t-x))e^{-\int_y^{y+t-x}\mu(\tau)d\tau}\Big]\,dy\right|\\
\le&\left\{\overline{\mu}\sup_{y\in[0,\infty)}\left|e^{-\int_y^{y+t-x-h}\mu(\tau)d\tau}-e^{-\int_y^{y+t-x}\mu(\tau)d\tau}\right|\right.\\
&\left.+\ |h|\sup_{y\in[0,\infty)}e^{-\int_y^{y+t-x}\mu(\tau)d\tau}\right\}\|\varphi_2\|_{L^1[0,\infty)}\\
\to&\ 0\quad\text{as}\quad |h|\to 0,\quad \text{uniformly for}\quad \varphi.
\end{aligned}
\tag{3-67}
$$

By using Proposition 3.5, (3-59), Lipschitz continuity of $\mu(x)$ and noting $t-x-h-y<t-x-h$ we estimate the second term of (3-64) as follows.

$$
\begin{aligned}
&\left|\int_0^{t-x-h}\mu(t-x-h-y)p_1(t-x-h-y,t-x-h)dy\right.\\
&\quad\left.-\int_0^{t-x}\mu(t-x-y)p_1(t-x-y,t-x)dy\right|\\
&=\left|\int_0^{t-x-h}\mu(t-x-h-y)p_1(t-x-h-y,t-x-h)dy\right.\\
&\quad-\int_0^{t-x}\mu(t-x-h-y)p_1(t-x-h-y,t-x-h)dy\\
&\quad+\int_0^{t-x}\mu(t-x-h-y)p_1(t-x-h-y,t-x-h)dy\\
&\quad\left.-\int_0^{t-x}\mu(t-x-y)p_1(t-x-y,t-x)dy\right|\\
&\le\int_{t-x-h}^{t-x}\mu(t-x-h-y)|p_1(t-x-h-y,t-x-h)|dy\\
&\quad+\int_0^{t-x}|\mu(t-x-h-y)p_1(t-x-h-y,t-x-h)\\
&\quad-\mu(t-x-y)p_1(t-x-y,t-x)|dy
\end{aligned}
$$

$$
\begin{aligned}
&= \int_{t-x-h}^{t-x} \mu(t-x-h-y)|p_1(0,y)| \\
&\quad \times e^{-\lambda\eta_1(t-x-h-y)-\int_0^{t-x-h-y}\mu(\tau)d\tau} dy \\
&\quad + \int_0^{t-x} \Big| \mu(t-x-h-y)p_1(0,y) \\
&\quad \times e^{-\lambda\eta_1(t-x-h-y)-\int_0^{t-x-h-y}\mu(\tau)d\tau} \\
&\quad - \mu(t-x-y)p_1(0,y)e^{-\lambda\eta_1(t-x-y)-\int_0^{t-x-y}\mu(\tau)d\tau} \Big| dy \\
&= \int_{t-x-h}^{t-x} |p_1(0,y)|\mu(t-x-h-y) \\
&\quad \times e^{-\lambda\eta_1(t-x-h-y)-\int_0^{t-x-h-y}\mu(\tau)d\tau} dy \\
&\quad + \int_0^{t-x} |p_1(0,y)| \\
&\quad \times \Big| \mu(t-x-h-y)e^{-\lambda\eta_1(t-x-h-y)-\int_0^{t-x-h-y}\mu(\tau)d\tau} \\
&\quad - \mu(t-x-y)e^{-\lambda\eta_1(t-x-y)-\int_0^{t-x-y}\mu(\tau)d\tau} \Big| dy \\
&\le \int_{t-x-h}^{t-x} |p_1(0,y)|\mu(t-x-h-y) \\
&\quad \times e^{-\lambda\eta_1(t-x-h-y)-\int_0^{t-x-h-y}\mu(\tau)d\tau} dy \\
&\quad + \int_0^{t-x} |p_1(0,y)||\mu(t-x-h-y)-\mu(t-x-y)| \\
&\quad \times e^{-\lambda\eta_1(t-x-h-y)-\int_0^{t-x-h-y}\mu(\tau)d\tau} dy \\
&\quad + \int_0^{t-x} |p_1(0,y)|\mu(t-x-y) \Big| e^{-\lambda\eta_1(t-x-h-y)-\int_0^{t-x-h-y}\mu(\tau)d\tau} \\
&\quad - e^{-\lambda\eta_1(t-x-y)-\int_0^{t-x-y}\mu(\tau)d\tau} \Big| dy \\
&\le \max\{\lambda\eta_1, \overline{\mu}q_2, \overline{\mu}\eta_2\}\|\varphi\| \\
&\quad \times \Bigg\{ |h|\overline{\mu} \sup_{y\in[0,\infty)} e^{-\lambda\eta_1(t-x-h-y)-\int_0^{t-x-h-y}\mu(\tau)d\tau} \\
&\quad + |h| \int_0^{t-x} e^{-\lambda\eta_1(t-x-h-y)-\int_0^{t-x-h-y}\mu(\tau)d\tau} dy \\
&\quad + \overline{\mu} \int_0^{t-x} \Big| e^{-\lambda\eta_1(t-x-h-y)-\int_0^{t-x-h-y}\mu(\tau)d\tau} \\
&\quad - e^{-\lambda\eta_1(t-x-y)-\int_0^{t-x-y}\mu(\tau)d\tau} \Big| dy \Bigg\} \\
&\quad \to 0 \quad \text{as} \quad |h| \to 0, \quad \text{uniformly for } \varphi.
\end{aligned}
\tag{3-68}
$$

By using Proposition 3.5, (3-62), (3-59) and Lipschitz continuity of $\mu(x)$, and noting $t-x-h-y<t-x-h$ we estimate the fourth term of (3-64) as follows.

$$
\begin{aligned}
&\left|\int_0^{t-x-h} \mu(t-x-h-y)p_2(t-x-h-y,t-x-h)dy\right.\\
&\quad\left.-\int_0^{t-x} \mu(t-x-y)p_2(t-x-y,t-x)dy\right|\\
&=\left|\int_0^{t-x-h} \mu(t-x-h-y)\left[p_2(0,y)e^{-\int_0^{t-x-h-y}\mu(\tau)d\tau}\right.\right.\\
&\quad\left.+\,p_1(0,y)e^{-\int_0^{t-x-h-y}\mu(\tau)d\tau}\left(1-e^{-\lambda\eta_1(t-x-h-y)}\right)\right]dy\\
&\quad-\int_0^{t-x} \mu(t-x-y)\left[p_2(0,y)e^{-\int_0^{t-x-y}\mu(\tau)d\tau}\right.\\
&\quad\left.\left.+\,p_1(0,y)e^{-\int_0^{t-x-y}\mu(\tau)d\tau}\left(1-e^{-\lambda\eta_1(t-x-y)}\right)\right]dy\right|\\
&\le\left|\int_0^{t-x-h} p_2(0,y)\mu(t-x-h-y)e^{-\int_0^{t-x-h-y}\mu(\tau)d\tau}dy\right.\\
&\quad\left.-\int_0^{t-x} p_2(0,y)\mu(t-x-y)e^{-\int_0^{t-x-y}\mu(\tau)d\tau}dy\right|\\
&\quad+\left|\int_0^{t-x-h} p_1(0,y)\mu(t-x-h-y)e^{-\int_0^{t-x-h-y}\mu(\tau)d\tau}\right.\\
&\quad\times\left(1-e^{-\lambda\eta_1(t-x-h-y)}\right)dy\\
&\quad-\int_0^{t-x} p_1(0,y)\mu(t-x-y)e^{-\int_0^{t-x-y}\mu(\tau)d\tau}\\
&\quad\left.\times\left(1-e^{-\lambda\eta_1(t-x-y)}\right)dy\right|\\
&\le\left|\int_0^{t-x-h} p_2(0,y)\mu(t-x-h-y)e^{-\int_0^{t-x-h-y}\mu(\tau)d\tau}dy\right.\\
&\quad\left.-\int_0^{t-x} p_2(0,y)\mu(t-x-h-y)e^{-\int_0^{t-x-h-y}\mu(\tau)d\tau}dy\right|\\
&\quad+\left|\int_0^{t-x} p_2(0,y)\mu(t-x-h-y)e^{-\int_0^{t-x-h-y}\mu(\tau)d\tau}dy\right.\\
&\quad\left.-\int_0^{t-x} p_2(0,y)\mu(t-x-y)e^{-\int_0^{t-x-y}\mu(\tau)d\tau}dy\right|\\
&\quad+\left|\int_0^{t-x-h} p_1(0,y)\mu(t-x-h-y)e^{-\int_0^{t-x-h-y}\mu(\tau)d\tau}\right.
\end{aligned}
$$

$$
\begin{aligned}
&\times \left(1 - e^{-\lambda\eta_1(t-x-h-y)}\right) dy \\
&- \int_0^{t-x} p_1(0,y)\mu(t-x-h-y)e^{-\int_0^{t-x-h-y}\mu(\tau)d\tau} \\
&\times \left(1 - e^{-\lambda\eta_1(t-x-h-y)}\right) dy \\
&+ \int_0^{t-x} p_1(0,y)\mu(t-x-h-y)e^{-\int_0^{t-x-h-y}\mu(\tau)d\tau} \\
&\times \left(1 - e^{-\lambda\eta_1(t-x-h-y)}\right) dy \\
&- \int_0^{t-x} p_1(0,y)\mu(t-x-y)e^{-\int_0^{t-x-y}\mu(\tau)d\tau} \\
&\times \left(1 - e^{-\lambda\eta_1(t-x-y)}\right) dy \Bigg| \\
= &\left| \int_{t-x-h}^{t-x} p_2(0,y)\mu(t-x-h-y)e^{-\int_0^{t-x-h-y}\mu(\tau)d\tau} dy \right| \\
&+ \Bigg| \int_0^{t-x} p_2(0,y) \Big[\mu(t-x-h-y)e^{-\int_0^{t-x-h-y}\mu(\tau)d\tau} \\
&-\mu(t-x-y)e^{-\int_0^{t-x-y}\mu(\tau)d\tau}\Big] dy \Bigg| \\
&+ \Bigg| \int_{t-x-h}^{t-x} p_1(0,y)\mu(t-x-h-y)e^{-\int_0^{t-x-h-y}\mu(\tau)d\tau} \\
&\times \left(1 - e^{-\lambda\eta_1(t-x-h-y)}\right) dy \Bigg| \\
&+ \Bigg| \int_0^{t-x} p_1(0,y) \Big[\mu(t-x-h-y)e^{-\int_0^{t-x-h-y}\mu(\tau)d\tau} \\
&\times \left(1 - e^{-\lambda\eta_1(t-x-h-y)}\right) \\
&- \mu(t-x-y)e^{-\int_0^{t-x-y}\mu(\tau)d\tau} \left(1 - e^{-\lambda\eta_1(t-x-y)}\right)\Big] dy \Bigg| \\
\le &\int_{t-x-h}^{t-x} |p_2(0,y)|\mu(t-x-h-y)e^{-\int_0^{t-x-h-y}\mu(\tau)d\tau} dy \\
&+ \int_0^{t-x} |p_2(0,y)||\mu(t-x-h-y) - \mu(t-x-y)| \\
&\times e^{-\int_0^{t-x-h-y}\mu(\tau)d\tau} dy \\
&+ \int_0^{t-x} |p_2(0,y)|\mu(t-x-y) \\
&\times \left| e^{-\int_0^{t-x-h-y}\mu(\tau)d\tau} - e^{-\int_0^{t-x-y}\mu(\tau)d\tau} \right| dy \\
&+ \int_{t-x-h}^{t-x} |p_1(0,y)|\mu(t-x-h-y)e^{-\int_0^{t-x-h-y}\mu(\tau)d\tau}
\end{aligned}
$$

$$
\begin{aligned}
&\times \left(1 - e^{-\lambda\eta_1(t-x-h-y)}\right) dy \\
&+ \int_0^{t-x} |p_1(0,y)||\mu(t-x-h-y) - \mu(t-x-y)| \\
&\times e^{-\int_0^{t-x-h-y}\mu(\tau)d\tau} \left(1 - e^{-\lambda\eta_1(t-x-h-y)}\right) dy \\
&+ \int_0^{t-x} |p_1(0,y)|\mu(t-x-y) \Big| e^{-\int_0^{t-x-h-y}\mu(\tau)d\tau} \\
&\times \left(1 - e^{-\lambda\eta_1(t-x-h-y)}\right) \\
&- e^{-\int_0^{t-x-y}\mu(\tau)d\tau} \left(1 - e^{-\lambda\eta_1(t-x-y)}\right) \Big| dy \\
\le\ & \overline{\mu} q_2 \|\varphi\| \Bigg\{ |h|\overline{\mu} \sup_{y\in[0,\infty)} e^{-\int_0^{t-x-h-y}\mu(\tau)d\tau} \\
&+ |h| \int_0^{t-x} e^{-\int_0^{t-x-h-y}\mu(\tau)d\tau} dy \\
&+ \overline{\mu} \int_0^{t-x} \left| e^{-\int_0^{t-x-h-y}\mu(\tau)d\tau} - e^{-\int_0^{t-x-y}\mu(\tau)d\tau} \right| dy \Bigg\} \\
&+ \max\{\lambda\eta_1, \overline{\mu} q_2, \overline{\mu}\eta_2\} \|\varphi\| \\
&\times \Bigg\{ \overline{\mu}|h| \sup_{y\in[0,\infty)} e^{-\int_0^{t-x-h-y}\mu(\tau)d\tau} \left(1 - e^{-\lambda\eta_1(t-x-h-y)}\right) \\
&+ |h| \int_0^{t-x} e^{-\int_0^{t-x-h-y}\mu(\tau)d\tau} \left(1 - e^{-\lambda\eta_1(t-x-h-y)}\right) dy \\
&+ \overline{\mu} \int_0^{t-x} \Big| e^{-\int_0^{t-x-h-y}\mu(\tau)d\tau} \left(1 - e^{-\lambda\eta_1(t-x-h-y)}\right) \\
&- e^{-\int_0^{t-x-y}\mu(\tau)d\tau} \left(1 - e^{-\lambda\eta_1(t-x-y)}\right) \Big| dy \Bigg\} \\
&\to 0 \quad \text{as} \quad |h| \to 0, \quad \text{uniformly for } \varphi. \qquad (3\text{-}69)
\end{aligned}
$$

By combining (3-69), (3-68), (3-67), (3-66) and (3-65) with (3-64) we deduce

$$
\begin{aligned}
&|p_1(0, t-x-h) - p_1(0, t-x)| \to 0 \\
&\qquad \text{as} \quad |h| \to 0, \quad \text{uniformly for} \quad \varphi. \qquad (3\text{-}70)
\end{aligned}
$$

Which implies that the second term and sixth term in (3-58) satisfy

$$
\begin{aligned}
&\int_0^t |p_1(0, t-x-h) - p_1(0, t-x)| e^{-\lambda\eta_1 x - \int_0^x \mu(\tau)d\tau} dx \\
&\to 0 \quad \text{as} \quad |h| \to 0, \quad \text{uniformly for } \varphi, \qquad (3\text{-}71)
\end{aligned}
$$

$$\int_0^t |p_1(0,t-x-h)-p_1(0,t-x)|e^{-\int_0^x \mu(\tau)d\tau}\left(1-e^{-\lambda\eta_1 x}\right)dx$$
$$\to 0 \quad \text{as} \quad |h|\to 0, \quad \text{uniformly for} \quad \varphi. \tag{3-72}$$

By using Proposition 3.5, (3-46), (3-59), (3-62), Lipschitz continuity of $\mu(x)$ and $t-x-h-y<t-x-h$, $t-x-y<t-x$ we have

$$\begin{aligned}
&|p_2(0,t-x-h)-p_2(0,t-x)| \\
&= \left|q_2\int_0^\infty \mu(s)p_2(s,t-x-h)ds - q_2\int_0^\infty \mu(s)p_2(s,t-x)ds\right| \\
&= q_2\left|\int_0^{t-x-h}\mu(s)p_2(s,t-x-h)ds\right. \\
&\quad + \int_{t-x-h}^\infty \mu(s)p_2(s,t-x-h)ds \\
&\quad \left.- \int_0^{t-x}\mu(s)p_2(s,t-x)ds - \int_{t-x}^\infty \mu(s)p_2(s,t-x)ds\right| \\
&\xlongequal[]{t-x-h-s=y,\ t-x-s=y} \\
&q_2\left|\int_0^{t-x-h}\mu(t-x-h-y)p_2(t-x-h-y,t-x-h)dy\right. \\
&\quad + \int_{t-x-h}^\infty \mu(s)\left[\varphi_2(s-t+x+h)e^{-\int_{s-t+x+h}^s \mu(\tau)d\tau}\right. \\
&\quad \left.+ \varphi_1(s-t+x+h)e^{-\int_{s-t+x+h}^s \mu(\tau)d\tau}\left(1-e^{-\lambda\eta_1(t-x-h)}\right)\right]ds \\
&\quad - \int_0^{t-x}\mu(t-x-y)p_2(t-x-y,t-x)dy \\
&\quad - \int_{t-x}^\infty \mu(s)\left[\varphi_2(s-t+x)e^{-\int_{s-t+x}^s \mu(\tau)d\tau}\right. \\
&\quad \left.\left. - \varphi_1(s-t+x)e^{-\int_{s-t+x}^s \mu(\tau)d\tau}\left(1-e^{-\lambda\eta_1(t-x)}\right)\right]ds\right| \\
&\le q_2\left|\int_0^{t-x-h}\mu(t-x-h-y)p_2(t-x-h-y,t-x-h)dy\right. \\
&\quad \left.- \int_0^{t-x}\mu(t-x-y)p_2(t-x-y,t-x)dy\right| \\
&\quad + q_2\left|\int_{t-x-h}^\infty \mu(s)\varphi_2(s-t+x+h)e^{-\int_{s-t+x+h}^s \mu(\tau)d\tau}ds\right. \\
&\quad \left.- \int_{t-x}^\infty \mu(s)\varphi_2(s-t+x)e^{-\int_{s-t+x}^s \mu(\tau)d\tau}ds\right|
\end{aligned}$$

$$
\begin{aligned}
&+ q_2 \left| \int_{t-x-h}^{\infty} \mu(s)\varphi_1(s-t+x+h) e^{-\int_{s-t+x+h}^{s} \mu(\tau)d\tau} \right. \\
&\quad \times \left(1 - e^{-\lambda\eta_1(t-x-h)}\right) ds \\
&- \int_{t-x}^{\infty} \mu(s)\varphi_1(s-t+x) e^{-\int_{s-t+x}^{s} \mu(\tau)d\tau} \\
&\quad \left. \times \left(1 - e^{-\lambda\eta_1(t-x)}\right) ds \right| \\
&\overset{s-t+x+h=y,\ s-t+x=y}{=\!=\!=\!=\!=} \\
&q_2 \left| \int_0^{t-x-h} \mu(t-x-h-y) \left[p_2(0,y) e^{-\int_0^{t-x-h-y} \mu(\tau)d\tau} \right.\right. \\
&\quad \left. + p_1(0,y) e^{-\int_0^{t-x-h-y} \mu(\tau)d\tau} \left(1 - e^{-\lambda\eta_1(t-x-h-y)}\right) \right] dy \\
&- \int_0^{t-x} \mu(t-x-y) \left[p_2(0,y) e^{-\int_0^{t-x-y} \mu(\tau)d\tau} \right. \\
&\quad \left.\left. + p_1(0,y) e^{-\int_0^{t-x-y} \mu(\tau)d\tau} \left(1 - e^{-\lambda\eta_1(t-x-y)}\right) \right] \right| dy \\
&+ q_2 \left| \int_0^{\infty} \mu(y+t-x-h)\varphi_2(y) e^{-\int_y^{y+t-x-h} \mu(\tau)d\tau} dy \right. \\
&\quad \left. + \int_0^{\infty} \mu(y+t-x)\varphi_2(y) e^{-\int_y^{y+t-x} \mu(\tau)d\tau} \right| \\
&+ q_2 \left| \int_0^{\infty} \varphi_1(y)\mu(y+t-x-h) e^{-\int_y^{y+t-x-h} \mu(\tau)d\tau} \right. \\
&\quad \times \left(1 - e^{-\lambda\eta_1(t-x-h)}\right) dy \\
&- \left. \int_0^{\infty} \varphi_1(y)\mu(y+t-x) e^{-\int_y^{y+t-x} \mu(\tau)d\tau} \left(1 - e^{-\lambda\eta_1(t-x)}\right) dy \right| \\
&\le q_2 \int_{t-x-h}^{t-x} \mu(t-x-h-y)|p_2(0,y)| e^{-\int_0^{t-x-h-y} \mu(\tau)d\tau} \\
&+ q_2 \int_0^{t-x} |p_2(0,y)||\mu(t-x-h-y) - \mu(t-x-y)| \\
&\quad \times e^{-\int_0^{t-x-h-y} \mu(\tau)d\tau} dy \\
&+ q_2 \int_0^{t-x} |p_2(0,y)|\mu(t-x-y) \\
&\times \left| e^{-\int_0^{t-x-h-y} \mu(\tau)d\tau} - e^{-\int_0^{t-x-y} \mu(\tau)d\tau} \right| dy \\
&+ q_2 \int_{t-x-h}^{t-x} |p_1(0,y)|\mu(t-x-h-y)
\end{aligned}
$$

$$
\begin{aligned}
&\times e^{-\int_0^{t-x-h-y}\mu(\tau)d\tau}\left(1-e^{-\lambda\eta_1(t-x-h-y)}\right)dy \\
&+ q_2\int_0^{t-x}|p_1(0,y)||\mu(t-x-h-y)-\mu(t-x-y)| \\
&\times e^{-\int_0^{t-x-h-y}\mu(\tau)d\tau}\left(1-e^{-\lambda\eta_1(t-x-h-y)}\right)dy \\
&+ q_2\int_0^{t-x}|p_1(0,y)|\mu(t-x-y)\left|e^{-\int_0^{t-x-h-y}\mu(\tau)d\tau}\right. \\
&\times\left(1-e^{-\lambda\eta_1(t-x-h-y)}\right) \\
&-\left. e^{-\int_0^{t-x-y}\mu(\tau)d\tau}(1-e^{-\lambda\eta_1(t-x-y)})\right|dy \\
&+ q_2\int_0^{\infty}|\varphi_2(y)||\mu(y+t-x-h)-\mu(y+t-x)| \\
&\times e^{-\int_y^{y+t-x-h}\mu(\tau)d\tau}dy \\
&+ q_2\int_0^{\infty}|\varphi_2(y)|\mu(y+t-x) \\
&\times\left|e^{-\int_y^{y+t-x-h}\mu(\tau)d\tau}-e^{-\int_y^{y+t-x}\mu(\tau)d\tau}\right|dy \\
&+ q_2\int_0^{\infty}|\varphi_1(y)||\mu(y+t-x-h)-\mu(y+t-x)| \\
&\times e^{-\int_y^{y+t-x-h}\mu(\tau)d\tau}\left(1-e^{-\lambda\eta_1(t-x-h)}\right)dy \\
&+ q_2\int_0^{\infty}|\varphi_1(y)|\mu(y+t-x) \\
&\times\left|e^{-\int_y^{y+t-x-h}\mu(\tau)d\tau}\left(1-e^{-\lambda\eta_1(t-x-h)}\right)\right. \\
&\left.-e^{-\int_y^{y+t-x}\mu(\tau)d\tau}\left(1-e^{-\lambda\eta_1(t-x)}\right)\right|dy \\
\le{}& q_2\overline{\mu}q_2\|\varphi\|\left\{\overline{\mu}|h|\sup_{y\in[0,\infty)}e^{-\int_0^{t-x-h-y}\mu(\tau)d\tau}\right. \\
&+|h|\int_0^{t-x}e^{-\int_0^{t-x-h-y}\mu(\tau)d\tau}dy \\
&\left.+\overline{\mu}\int_0^{t-x}\left|e^{-\int_0^{t-x-y-h}\mu(\tau)d\tau}-e^{-\int_0^{t-x-y}\mu(\tau)d\tau}\right|dy\right\} \\
&+q_2\max\{\lambda\eta_1,\overline{\mu}\eta_2,\overline{\mu}q_2\}\|\varphi\| \\
&\times\left\{\overline{\mu}|h|\sup_{y\in[0,\infty)}e^{-\int_0^{t-x-h-y}\mu(\tau)d\tau}\left(1-e^{-\lambda\eta_1(t-x-h-y)}\right)\right. \\
&+|h|\int_0^{t-x}e^{-\int_0^{t-x-h-y}\mu(\tau)d\tau}\left(1-e^{-\lambda\eta_1(t-x-h-y)}\right)dy
\end{aligned}
$$

$$
\begin{aligned}
&+\overline{\mu}\int_0^{t-x}\Big|e^{-\int_0^{t-x-h-y}\mu(\tau)d\tau}\left(1-e^{-\lambda\eta_1(t-x-h-y)}\right)\\
&-\ e^{-\int_0^{t-x-y}\mu(\tau)d\tau}\left(1-e^{-\lambda\eta_1(t-x-y)}\right)\Big|\,dy\Big\}\\
&+q_2\|\varphi_2\|_{L^1[0,\infty)}\left\{|h|\sup_{y\in[0,\infty)}e^{-\int_y^{y+t-x-h}\mu(\tau)d\tau}\right.\\
&\left.+\overline{\mu}\sup_{y\in[0,\infty)}\left|e^{-\int_y^{y+t-x-h}\mu(\tau)d\tau}-e^{-\int_y^{y+t-x}\mu(\tau)d\tau}\right|\right\}\\
&+\Big\{|h|\sup_{y\in[0,\infty)}e^{-\int_y^{y+t-x-h}\mu(\tau)d\tau}\left(1-e^{-\lambda\eta_1(t-x-h)}\right)\\
&+\overline{\mu}\sup_{y\in[0,\infty)}\Big|e^{-\int_y^{y+t-x-h}\mu(\tau)d\tau}\left(1-e^{-\lambda\eta_1(t-x-h)}\right)\\
&-\ e^{-\int_y^{y+t-x}\mu(\tau)d\tau}\left(1-e^{-\lambda\eta_1(t-x)}\right)\Big|\Big\}\,q_2\|\varphi_1\|_{L^1[0,\infty)}\\
&\to 0\quad\text{as}\quad |h|\to 0,\quad\text{uniformly for }\varphi. \qquad (3\text{-}73)
\end{aligned}
$$

Which implies that the fourth term in (3-58) satisfies

$$
\begin{aligned}
&\int_0^t|p_2(0,t-x-h)-p_2(0,t-x)|e^{-\int_0^x\mu(\tau)d\tau}dx\to 0,\\
&\quad\text{as }|h|\to 0\text{ uniformly for }\varphi. \qquad (3\text{-}74)
\end{aligned}
$$

By combining (3-71), (3-72), (3-74), (3-60), (3-61) and (3-63) with (3-58) we deduce, for $x+h\in[0,t)$,

$$
\begin{aligned}
&\sum_{i=1}^{2}\int_0^t|p_i(x+h,t)-p_i(x,t)|dx\to 0\\
&\quad\text{as}\quad |h|\to 0,\quad\text{uniformly for }\varphi. \qquad (3\text{-}75)
\end{aligned}
$$

If $h\in(-t,0),\ x\in[0,t)$, then $p_1(x+h,t)=0$ and $p_2(x+h,t)=0$ for $x+h<0$ give

$$
\begin{aligned}
&\int_0^t|p_1(x+h,t)-p_1(x,t)|dx\\
&\quad+\int_0^t|p_2(x+h,t)-p_2(x,t)|dx\\
&=\int_{-h}^t|p_1(x+h,t)-p_1(x,t)|dx\\
&\quad+\int_0^{-h}|p_1(x+h,t)-p_1(x,t)|dx\\
&\quad+\int_{-h}^t|p_2(x+h,t)-p_2(x,t)|dx
\end{aligned}
$$

$$
\begin{aligned}
&+\int_0^{-h} |p_2(x+h,t)-p_2(x,t)|dx \\
&= \int_{-h}^{t} |p_1(x+h,t)-p_1(x,t)|dx + \int_0^{-h} |p_1(x,t)|dx \\
&+\int_{-h}^{t} |p_2(x+h,t)-p_2(x,t)|dx + \int_0^{-h} |p_2(x,t)|dx. \qquad (3\text{-}76)
\end{aligned}
$$

Since $x+h \in [0,t)$ for $x \in [0,t)$, $h \in [-t,0)$, similar way to (3-75) we have, for the first term and third term in (3-76),

$$
\int_{-h}^{t} |p_1(x+h,t)-p_1(x,t)|dx \to 0,
$$
$$
\text{as } |h| \to 0, \text{ uniformly for } \varphi. \qquad (3\text{-}77)
$$
$$
\int_{-h}^{t} |p_2(x+h,t)-p_2(x,t)|dx \to 0,
$$
$$
\text{as } |h| \to 0, \text{ uniformly for } \varphi. \qquad (3\text{-}78)
$$

By using Proposition 3.5 and (3-59) we estimate the second term in (3-76) as follows:

$$
\begin{aligned}
\int_0^{-h} |p_1(x,t)|dx &= \int_0^{-h} |p_1(0,t-x)| e^{-\lambda\eta_1 x - \int_0^x \mu(\tau)d\tau} dx \\
&\le \max\{\lambda\eta_1, \overline{\mu}\eta_2, \overline{\mu}q_2\}\|\varphi\| \int_0^{-h} e^{-\lambda\eta_1 x - \int_0^x \mu(\tau)d\tau} dx \\
&\to 0, \quad \text{as} \quad |h| \to 0, \text{ uniformly for } \varphi. \qquad (3\text{-}79)
\end{aligned}
$$

It is the same as (3-79), by using Proposition 3.5, (3-59) and (3-62) we deduce

$$
\int_0^{-h} |p_2(x,t)|dx \to 0, \text{ as } |h| \to 0, \text{ uniformly for } \varphi. \qquad (3\text{-}80)
$$

Combining (3-77)–(3-80) with (3-76) we obtain, for $h \in (-t,0)$,

$$
\sum_{i=1}^{2} \int_0^t |p_i(x+h,t)-p_i(x,t)|dx \to 0
$$
$$
\text{as } |h| \to 0, \text{ uniformly for } \varphi. \qquad (3\text{-}81)
$$

(3-81) and (3-75) show that the result of this theorem is true. □

Theorem 3.7. *Assume that there exist two positive constants $\overline{\mu}$ and $\underline{\mu}$ such that $0 < \underline{\mu} \le \mu(x) \le \overline{\mu}$, then $V(t)$ satisfies*

$$
\|V(t)\varphi\| \le e^{-\min\{\lambda\eta_1, \underline{\mu}\}} \|\varphi\|, \quad \forall \varphi \in X. \qquad (3\text{-}82)
$$

Proof. For any $\varphi \in X$ from the definition of $V(t)$ and Proposition 3.5 we have

$$
\begin{aligned}
\|V(t)\varphi(\cdot)\| &= |\varphi_0 e^{-\lambda\eta_1 t}| \\
&\quad + \int_t^\infty \left|\varphi_1(x-t)e^{-\lambda\eta_1 t-\int_{x-t}^x \mu(\tau)d\tau}\right| dx \\
&\quad + \int_t^\infty \Big|\varphi_2(x-t)e^{-\int_{x-t}^x \mu(\tau)d\tau} \\
&\quad +\varphi_1(x-t)e^{-\int_{x-t}^x \mu(\tau)d\tau}\left(1-e^{-\lambda\eta_1 t}\right)\Big| dx \\
&\le |\varphi_0|e^{-\lambda\eta_1 t} \\
&\quad + \sup_{x\in[t,\infty)} \left|e^{-\lambda\eta_1 t-\int_{x-t}^x \mu(\tau)d\tau}\right| \int_t^\infty |\varphi_1(x-t)|dx \\
&\quad + \sup_{x\in[t,\infty)} \left|e^{-\int_{x-t}^x \mu(\tau)d\tau}\right| \int_t^\infty |\varphi_2(x-t)|dx \\
&\quad + \sup_{x\in[t,\infty)} e^{-\int_{x-t}^x \mu(\tau)d\tau}\left(1-e^{-\lambda\eta_1 t}\right)\int_t^\infty |\varphi_1(x-t)|dx \\
&\le |\varphi_0|e^{-\lambda\eta_1 t} + e^{-\lambda\eta_1 t-\underline{\mu}t}\int_t^\infty |\varphi_1(x-t)|dx \\
&\quad + e^{-\underline{\mu}t}\int_t^\infty |\varphi_2(x-t)|dx \\
&\quad + e^{-\underline{\mu}t}\left(1-e^{-\lambda\eta_1 t}\right)\int_t^\infty |\varphi_1(x-t)|dx \\
&\overset{x-t=\tau}{=\!=\!=} |\varphi_0|e^{-\lambda\eta_1 t} + e^{-(\lambda\eta_1+\underline{\mu})t}\int_0^\infty |\varphi_1(\tau)|d\tau \\
&\quad + e^{-\underline{\mu}t}\int_0^\infty |\varphi_2(\tau)|d\tau \\
&\quad + \left(1-e^{-\lambda\eta_1 t}\right)e^{-\underline{\mu}t}\int_0^\infty |\varphi_1(\tau)|d\tau \\
&= |\varphi_0|e^{-\lambda\eta_1 t} + e^{-(\lambda\eta_1+\underline{\mu})t}\|\varphi_1\|_{L^1[0,\infty)} \\
&\quad + e^{-\underline{\mu}t}\|\varphi_2\|_{L^1[0,\infty)} + \left(e^{-\underline{\mu}t} - e^{-(\lambda\eta_1+\underline{\mu})t}\right)\|\varphi_1\|_{L^1[0,\infty)} \\
&= |\varphi_0|e^{-\lambda\eta_1 t} + e^{-\underline{\mu}t}\|\varphi_1\|_{L^1[0,\infty)} + e^{-\underline{\mu}t}\|\varphi_2\|_{L^1[0,\infty)} \\
&\le e^{-\min\{\lambda\eta_1,\underline{\mu}\}t}\left(|\varphi_0| + \|\varphi_1\|_{L^1[0,\infty)} + \|\varphi_2\|_{L^1[0,\infty)}\right) \\
&= e^{-\min\{\lambda\eta_1,\underline{\mu}\}t}\|\varphi\|. \qquad (3\text{-}83)
\end{aligned}
$$

Which shows that the result of this theorem holds. □

From Theorem 3.6 and Theorem 3.7 we deduce

$$\|S(t)-W(t)\| = \|V(t)\| \le e^{-\min\{\lambda\eta_1,\underline{\mu}\}t} \to 0 \quad \text{as} \quad t\to\infty.$$

From which together with Definition 1.85 we derive the following result:

Theorem 3.8. *Assume that $\mu(x)$ is Lipschitz continuous and there exist two positive constants $\overline{\mu}$ and $\underline{\mu}$ such that $0 < \underline{\mu} \leq \mu(x) \leq \overline{\mu} < \infty$; then $S(t)$ is a quasi-compact operator on X.*

Since $E : X \to \mathbb{R}^3$ is a bounded linear operator, E is a compact operator on X and therefore by Theorem 3.8, and Proposition 2.9 in page 215 of Nagel [91] we conclude

Corollary 3.9. *Assume that $\mu(x)$ is Lipschitz continuous and there exist two positive constants $\overline{\mu}$ and $\underline{\mu}$ such that $0 < \underline{\mu} \leq \mu(x) \leq \overline{\mu} < \infty$, then $T(t)$ is a quasi-compact operator on X.*

Lemma 3.10. *0 is an eigenvalue of $A + U + E$ with geometric multiplicity 1.*

Proof. Consider the equation $(A + U + E)p = 0$, i.e,

$$\lambda\eta_1 p_0 = \eta_2 \int_0^\infty \mu(x)p_1(x)dx, \tag{3-84}$$

$$\frac{dp_1(x)}{dx} = -(\lambda\eta_1 + \mu(x))p_1(x), \tag{3-85}$$

$$\frac{dp_2(x)}{dx} = -\mu(x)p_2(x) + \lambda\eta_1 p_1(x), \tag{3-86}$$

$$\begin{aligned} p_1(0) = \lambda\eta_1 p_0 + q_2 \int_0^\infty \mu(x)p_1(x)dx \\ + \eta_2 \int_0^\infty \mu(x)p_2(x)dx, \end{aligned} \tag{3-87}$$

$$p_2(0) = q_2 \int_0^\infty \mu(x)p_2(x)dx. \tag{3-88}$$

By solving (3-85) and (3-86) we have

$$p_1(x) = a_1 e^{-\lambda\eta_1 x - \int_0^x \mu(\tau)d\tau}, \tag{3-89}$$

$$\begin{aligned} p_2(x) &= a_2 e^{-\int_0^x \mu(\tau)d\tau} \\ &\quad + \lambda\eta_1 e^{-\int_0^x \mu(\tau)d\tau} \int_0^x p_1(\xi) e^{\int_0^\xi \mu(\tau)d\tau} d\xi. \end{aligned} \tag{3-90}$$

Through inserting (3-89) into (3-84) we obtain

$$a_1 = \frac{\lambda\eta_1}{\eta_2 \int_0^\infty \mu(x) e^{-\lambda\eta_1 x - \int_0^x \mu(\tau)d\tau} dx} p_0. \tag{3-91}$$

By combining (3-90) and (3-89) with (3-88), and using the Fubini theorem (see Theorem 1.49), (3-91) and $q_2 + \eta_2 = 1$ we deduce

$$a_2 = q_2 \int_0^\infty \mu(x) \Big[a_2 e^{-\int_0^x \mu(\tau)d\tau}$$

$$
\begin{aligned}
&\quad +\lambda\eta_1 e^{-\int_0^x \mu(\tau)d\tau}\int_0^x p_1(\xi)e^{\int_0^\xi \mu(\tau)d\tau}d\xi\Big]\,dx \\
&= a_2q_2\int_0^\infty \mu(x)e^{-\int_0^x \mu(\tau)d\tau}dx \\
&\quad + \lambda\eta_1 q_2\int_0^\infty \mu(x)e^{-\int_0^x \mu(\tau)d\tau}\int_0^x p_1(\xi)e^{\int_0^\xi \mu(\tau)d\tau}d\xi dx \\
&= a_2q_2\left[-e^{-\int_0^x \mu(\tau)d\tau}\Big|_0^\infty\right] \\
&\quad + \lambda\eta_1 q_2\int_0^\infty p_1(\xi)e^{\int_0^\xi \mu(\tau)d\tau}\int_\xi^\infty \mu(x)e^{-\int_0^x \mu(\tau)d\tau}dxd\xi \\
&= a_2q_2 + \lambda\eta_1 q_2\int_0^\infty p_1(\xi)d\xi \\
&= a_2q_2 + \lambda\eta_1 q_2\int_0^\infty a_1 e^{-\lambda\eta_1\xi-\int_0^\xi \mu(\tau)d\tau}d\xi \\
&= a_2q_2 + a_1\lambda\eta_1 q_2\int_0^\infty e^{-\lambda\eta_1 x-\int_0^x \mu(\tau)d\tau}dx \\
&\Longrightarrow
\end{aligned}
$$

$$
\begin{aligned}
(1-q_2)a_2 &= \frac{\lambda\eta_1}{\eta_2\int_0^\infty \mu(x)e^{-\lambda\eta_1 x-\int_0^x \mu(\tau)d\tau}dx}p_0 \\
&\quad \times \lambda\eta_1 q_2\int_0^\infty e^{-\lambda\eta_1 x-\int_0^x \mu(\tau)d\tau}dx \\
&\Longrightarrow
\end{aligned}
$$

$$
a_2 = \frac{(\lambda\eta_1)^2 q_2\int_0^\infty e^{-\lambda\eta_1 x-\int_0^x \mu(\tau)d\tau}dx}{\eta_2^2\int_0^\infty \mu(x)e^{-\lambda\eta_1 x-\int_0^x \mu(\tau)d\tau}dx}p_0. \tag{3-92}
$$

From (3-89), (3-90), (3-91), (3-92) and the Fubini theorem (see Theorem 1.49) we estimate

$$
\begin{aligned}
\|p\| &= |p_0| + \|p_1\|_{L^1[0,\infty)} + \|p_2\|_{L^1[0,\infty)} \\
&\le |p_0| + |a_1|\int_0^\infty e^{-\lambda\eta_1 x-\int_0^x \mu(\tau)d\tau}dx \\
&\quad + |a_2|\int_0^\infty e^{-\int_0^x \mu(\tau)d\tau}dx \\
&\quad + \lambda\eta_1\int_0^\infty e^{-\int_0^x \mu(\tau)d\tau}\int_0^x |p_1(\xi)|e^{\int_0^\xi \mu(\tau)d\tau}d\xi dx \\
&= |p_0| + |a_1|\int_0^\infty e^{-\lambda\eta_1 x-\int_0^x \mu(\tau)d\tau}dx \\
&\quad + \frac{(\lambda\eta_1)^2 q_2\left(\int_0^\infty e^{-\lambda\eta_1 x-\int_0^x \mu(\tau)d\tau}dx\right)\int_0^\infty e^{-\int_0^x \mu(\tau)d\tau}}{\eta_2^2\int_0^\infty \mu(x)e^{-\lambda\eta_1 x-\int_0^x \mu(\tau)d\tau}dx}|p_0|
\end{aligned}
$$

$$
\begin{aligned}
&+ \lambda\eta_1 \int_0^\infty e^{-\int_0^x \mu(\tau)d\tau} \int_0^x |a_1| e^{-\lambda\eta_1\xi - \int_0^\xi \mu(\tau)d\tau} e^{\int_0^\xi \mu(\tau)d\tau} d\xi dx \\
= &|p_0| + |a_1| \int_0^\infty e^{-\lambda\eta_1 x - \int_0^x \mu(\tau)d\tau} dx \\
&+ \frac{(\lambda\eta_1)^2 q_2 \left(\int_0^\infty e^{-\lambda\eta_1 x - \int_0^x \mu(\tau)d\tau} dx\right) \int_0^\infty e^{-\int_0^x \mu(\tau)d\tau}}{\eta_2^2 \int_0^\infty \mu(x) e^{-\lambda\eta_1 x - \int_0^x \mu(\tau)d\tau} dx} |p_0| \\
&+ |a_1| \int_0^\infty e^{-\int_0^x \mu(\tau)d\tau} \left(1 - e^{-\lambda\eta_1 x}\right) dx \\
= &|p_0| + |a_1| \int_0^\infty e^{-\int_0^x \mu(\tau)d\tau} dx \\
&+ \frac{(\lambda\eta_1)^2 q_2 \left(\int_0^\infty e^{-\lambda\eta_1 x - \int_0^x \mu(\tau)d\tau} dx\right) \int_0^\infty e^{-\int_0^x \mu(\tau)d\tau}}{\eta_2^2 \int_0^\infty \mu(x) e^{-\lambda\eta_1 x - \int_0^x \mu(\tau)d\tau} dx} |p_0| \\
= &|p_0| + \frac{\lambda\eta_1 \int_0^\infty e^{-\int_0^x \mu(\tau)d\tau} dx}{\eta_2 \int_0^\infty \mu(x) e^{-\lambda\eta_1 x - \int_0^x \mu(\tau)d\tau} dx} |p_0| \\
&+ \frac{(\lambda\eta_1)^2 q_2 \left(\int_0^\infty e^{-\lambda\eta_1 x - \int_0^x \mu(\tau)d\tau} dx\right) \int_0^\infty e^{-\int_0^x \mu(\tau)d\tau}}{\eta_2^2 \int_0^\infty \mu(x) e^{-\lambda\eta_1 x - \int_0^x \mu(\tau)d\tau} dx} |p_0| \\
< &\infty.
\end{aligned}
$$

This together with Definition 1.22 show that 0 is an eigenvalue of $A + U + E$. Moreover, by (3-89), (3-90), (3-91) and (3-92) it is immediately seen that the eigenvector space corresponding to 0 is a one-dimensional linear space, that is, the geometric multiplicity of 0 is 1 (see Definition 1.23). □

It is easy to verify that X^*, the dual space of X, is as follows (see Theorem 1.20).

$$
X^* = \left\{ q^* \;\middle|\; \begin{array}{l} q^* \in \mathbb{R} \times L^\infty[0,\infty) \times L^\infty[0,\infty), \\ |||q^*||| = \max\{|q_0^*|, \|q_1^*\|_{L^\infty[0,\infty)}, \|q_2^*\|_{L^\infty[0,\infty)}\} \end{array} \right\}.
$$

It is obvious that X^* is a Banach space.

Lemma 3.11. $(A + U + E)^*$,*adjoint operator of* $A + U + E$, *is as follows.*

$$
(A + U + E)^* q^* = (G + F) q^*, \quad \forall q^* \in D(G),
$$

where

$$
\begin{aligned}
Gq^*(x) = &\begin{pmatrix} -\lambda\eta_1 & 0 & 0 \\ 0 & \frac{d}{dx} - (\lambda\eta_1 + \mu(x)) & 0 \\ 0 & 0 & \frac{d}{dx} - \mu(x) \end{pmatrix} \begin{pmatrix} q_0^* \\ q_1^*(x) \\ q_2^*(x) \end{pmatrix} \\
&+ \begin{pmatrix} 0 & 0 & 0 \\ \mu(x)\eta_2 & \mu(x) q_2 & 0 \\ 0 & \mu(x)\eta_2 & \mu(x) q_2 \end{pmatrix} \begin{pmatrix} q_0^* \\ q_1^*(0) \\ q_2^*(0) \end{pmatrix},
\end{aligned}
$$

$$D(G) = \left\{ q^* \in X^* \;\middle|\; \frac{dq_i^*(x)}{dx} \text{ exist and } q_i^*(\infty) = \alpha,\ i = 1, 2 \right\};$$

$$\begin{aligned} Fq^*(x) = &\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \lambda\eta_1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} q_0^* \\ q_1^*(x) \\ q_2^*(x) \end{pmatrix} \\ &+ \begin{pmatrix} 0 & \lambda\eta_1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} q_0^* \\ q_1^*(0) \\ q_2^*(0) \end{pmatrix},\ D(F) = X^*, \end{aligned}$$

where α in $D(G)$ is a constant which is irrelevant to i.

Proof. By applying integration by parts and the boundary conditions on $p \in D(A)$ and $q^* \in D(G)$, we have

$$\begin{aligned} &\langle (A + U + E)p, q^* \rangle \\ &= \left\{ -\lambda\eta_1 p_0 + \eta_2 \int_0^\infty \mu(x) p_1(x) dx \right\} q_0^* \\ &\quad + \int_0^\infty \left\{ -\frac{dp_1(x)}{dx} - (\lambda\eta_1 + \mu(x)) p_1(x) \right\} q_1^*(x) dx \\ &\quad + \int_0^\infty \left\{ -\frac{dp_2(x)}{dx} - \mu(x) p_2(x) + \lambda\eta_1 p_1(x) \right\} q_2^*(x) dx \\ &= -\lambda\eta_1 p_0 q_0^* + \eta_2 q_0^* \int_0^\infty \mu(x) p_1(x) dx \\ &\quad - \int_0^\infty \frac{dp_1(x)}{dx} q_1^*(x) dx - \int_0^\infty (\lambda\eta_1 + \mu(x)) p_1(x) q_1^*(x) dx \\ &\quad - \int_0^\infty \frac{dp_2(x)}{dx} q_2^*(x) dx - \int_0^\infty \mu(x) p_2(x) q_2^*(x) dx \\ &\quad + \int_0^\infty \lambda\eta_1 p_1(x) q_2^*(x) dx \\ &= -\lambda\eta_1 p_0 q_0^* + \eta_2 q_0^* \int_0^\infty \mu(x) p_1(x) dx \\ &\quad - p_1(x) q_1^*(x) \Big|_0^\infty + \int_0^\infty p_1(x) \frac{dq_1^*(x)}{dx} dx \\ &\quad - \int_0^\infty (\lambda\eta_1 + \mu(x)) p_1(x) q_1^*(x) dx \\ &\quad - p_2(x) q_2^*(x) \Big|_0^\infty + \int_0^\infty p_2(x) \frac{dq_2^*(x)}{dx} dx \\ &\quad - \int_0^\infty \mu(x) p_2(x) q_2^*(x) dx + \int_0^\infty \lambda\eta_1 p_1(x) q_2^*(x) dx \end{aligned}$$

$$
\begin{aligned}
&= -\lambda\eta_1 p_0 q_0^* + \eta_2 q_0^* \int_0^\infty \mu(x) p_1(x) dx + p_1(0) q_1^*(0) \\
&\quad + \int_0^\infty p_1(x) \left\{ \frac{dq_1^*(x)}{dx} - (\lambda\eta_1 + \mu(x)) q_1^*(x) \right\} dx \\
&\quad + p_2(0) q_2^*(0) + \int_0^\infty p_2(x) \left\{ \frac{dq_2^*(x)}{dx} - \mu(x) q_2^*(x) \right\} dx \\
&\quad + \int_0^\infty \lambda\eta_1 p_1(x) q_2^*(x) dx \\
&= -\lambda\eta_1 p_0 q_0^* + \eta_2 q_0^* \int_0^\infty \mu(x) p_1(x) dx \\
&\quad + q_1^*(0) \left\{ \lambda\eta_1 p_0 + q_2 \int_0^\infty \mu(x) p_1(x) dx \right. \\
&\quad \left. + \eta_2 \int_0^\infty \mu(x) p_2(x) dx \right\} \\
&\quad + \int_0^\infty p_1(x) \left\{ \frac{dq_1^*(x)}{dx} - (\lambda\eta_1 + \mu(x)) q_1^*(x) \right\} dx \\
&\quad + q_2^*(0) q_2 \int_0^\infty \mu(x) p_2(x) dx \\
&\quad + \int_0^\infty p_2(x) \left\{ \frac{dq_2^*(x)}{dx} - \mu(x) q_2^*(x) \right\} dx \\
&\quad + \int_0^\infty \lambda\eta_1 p_1(x) q_2^*(x) dx \\
&= p_0(-\lambda\eta_1 q_0^* + \lambda\eta_1 q_1^*(0)) \\
&\quad + \int_0^\infty p_1(x) \left\{ \frac{dq_1^*(x)}{dx} - (\lambda\eta_1 + \mu(x)) q_1^*(x) \right. \\
&\quad \left. + \mu(x) q_2 q_1^*(0) + \mu(x) \eta_2 q_0^* + \lambda\eta_1 q_2^*(x) \right\} dx \\
&\quad + \int_0^\infty p_2(x) \left\{ \frac{dq_2^*(x)}{dx} - \mu(x) q_2^*(x) \right. \\
&\quad \left. + \mu(x) q_2 q_2^*(0) + \mu(x) \eta_2 q_1^*(0) \right\} dx \\
&= p_0(-\lambda\eta_1 q_0^*) \\
&\quad + \int_0^\infty p_1(x) \left\{ \frac{dq_1^*(x)}{dx} - (\lambda\eta_1 + \mu(x)) q_1^*(x) \right. \\
&\quad \left. + \mu(x) q_2 q_1^*(0) + \mu(x) \eta_2 q_0^* \right\} dx \\
&\quad + \int_0^\infty p_2(x) \left\{ \frac{dq_2^*(x)}{dx} - \mu(x) q_2^*(x) \right.
\end{aligned}
$$

$$\begin{aligned}
&+ \mu(x)\eta_2 q_1^*(0) + \mu(x) q_2 q_2^*(0)\Big\}dx \\
&+ p_0(\lambda\eta_1 q_1^*(0)) + \int_0^\infty p_1(x)\lambda\eta_1 q_2^*(x)dx \\
= &\langle p, (G+F)q^*\rangle. \qquad (3\text{-}93)
\end{aligned}$$

From which together with Definition 1.21 we know that the result of this lemma is right. ☐

Lemma 3.12. 0 *is an eigenvalue of* $(A+U+E)^*$ *with geometric multiplicity* 1.

Proof. Consider the equation $(A+U+E)^*q^* = (G+F)q^* = 0$. It is equivalent to

$$-\lambda\eta_1 q_0^* + \lambda\eta_1 q_1^*(0) = 0, \qquad (3\text{-}94)$$

$$\begin{aligned}
\frac{dq_1^*(x)}{dx} &= (\lambda\eta_1 + \mu(x))q_1^*(x) - \mu(x)\eta_2 q_0^* \\
&\quad - \mu(x)q_2 q_1^*(0) - \lambda\eta_1 q_2^*(x), \qquad (3\text{-}95)
\end{aligned}$$

$$\frac{dq_2^*(x)}{dx} = \mu(x)q_2^*(x) - \mu(x)\eta_2 q_1^*(0) - \mu(x)q_2 q_2^*(0), \qquad (3\text{-}96)$$

$$q_1^*(\infty) = q_2^*(\infty) = \alpha. \qquad (3\text{-}97)$$

By (3-94)–(3-96) we deduce

$$q_1^*(0) = q_0^*, \qquad (3\text{-}98)$$

$$\begin{aligned}
q_1^*(x) &= b_1 e^{\lambda\eta_1 x + \int_0^x \mu(\tau)d\tau} - e^{\lambda\eta_1 x + \int_0^x \mu(\tau)d\tau}\int_0^x [\mu(\tau)\eta_2 q_0^* \\
&\quad + \mu(\tau)q_2 q_1^*(0) + \lambda\eta_1 q_2^*(\tau)]e^{-\lambda\eta_1\tau - \int_0^\tau \mu(\xi)d\xi}d\tau, \qquad (3\text{-}99)
\end{aligned}$$

$$\begin{aligned}
q_2^*(x) &= b_2 e^{\int_0^x \mu(\tau)d\tau} - e^{\int_0^x \mu(\tau)d\tau}\int_0^x [\mu(\tau)\eta_2 q_1^*(0) \\
&\quad + \mu(\tau)q_2 q_2^*(0)]e^{-\int_0^\tau \mu(\xi)d\xi}d\tau. \qquad (3\text{-}100)
\end{aligned}$$

Through multiplying $e^{-\lambda\eta_1 x - \int_0^x \mu(\tau)d\tau}$ to two side of (3-99) and multiplying $e^{-\int_0^x \mu(\tau)d\tau}$ to two side of (3-100), and using (3-97) we obtain

$$\begin{aligned}
b_1 &= \int_0^\infty [\mu(\tau)\eta_2 q_0^* + \mu(\tau)q_2 q_1^*(0) \\
&\quad + \lambda\eta_1 q_2^*(\tau)]e^{-\lambda\eta_1\tau - \int_0^\tau \mu(\xi)d\xi}d\tau, \qquad (3\text{-}101)
\end{aligned}$$

$$b_2 = \int_0^\infty [\mu(\tau)\eta_2 q_1^*(0) + \mu(\tau)q_2 q_2^*(0)]e^{-\int_0^\tau \mu(\xi)d\xi}d\tau. \qquad (3\text{-}102)$$

By inserting (3-101) into (3-99) and (3-102) into (3-100) we have

$$q_1^*(x) = e^{\lambda\eta_1 x + \int_0^x \mu(\tau)d\tau}\int_x^\infty [\mu(\tau)\eta_2 q_0^*$$

$$+\mu(\tau)q_2q_1^*(0)+\lambda\eta_1q_2^*(\tau)]e^{-\lambda\eta_1\tau-\int_0^\tau\mu(\xi)d\xi}d\tau, \tag{3-103}$$

$$\begin{aligned}
q_2^*(x) &= e^{\int_0^x\mu(\tau)d\tau}\int_x^\infty[\mu(\tau)\eta_2q_1^*(0)+\mu(\tau)q_2q_2^*(0)]e^{-\int_0^\tau\mu(\xi)d\xi}d\tau\\
&= e^{\int_0^x\mu(\tau)d\tau}\left[\eta_2q_1^*(0)\int_x^\infty\mu(\tau)e^{-\int_0^\tau\mu(\xi)d\xi}d\tau\right.\\
&\quad\left.+q_2q_2^*(0)\int_x^\infty\mu(\tau)e^{-\int_0^\tau\mu(\xi)d\xi}d\tau\right]\\
&= e^{\int_0^x\mu(\tau)d\tau}\left[\eta_2q_1^*(0)\left(-e^{-\int_0^\tau\mu(\xi)d\xi}\Big|_x^\infty\right)\right.\\
&\quad\left.+q_2q_2^*(0)\left(-e^{-\int_0^\tau\mu(\xi)d\xi}\Big|_x^\infty\right)\right]\\
&= e^{\int_0^x\mu(\tau)d\tau}\left[\eta_2q_1^*(0)e^{-\int_0^x\mu(\xi)d\xi}+q_2q_2^*(0)e^{-\int_0^x\mu(\xi)d\xi}\right]\\
&= \eta_2q_1^*(0)+q_2q_2^*(0).
\end{aligned}\tag{3-104}$$

By applying the relation $\eta_2+q_2=1$ in (3-104) it follows that

$$q_2^*(0)=q_1^*(0). \tag{3-105}$$

By combining (3-105) with (3-104) we derive

$$q_2^*(x)=q_1^*(0)=q_2^*(0),\quad \forall x\in[0,\infty). \tag{3-106}$$

Through inserting (3-106) into (3-103) and using (3-98) and $\eta_2+q_2=1$ we deduce

$$\begin{aligned}
q_1^*(x) &= e^{\lambda\eta_1x+\int_0^x\mu(\tau)d\tau}\int_x^\infty[\mu(\tau)\eta_2q_0^*\\
&\quad+\mu(\tau)q_2q_0^*+\lambda\eta_1q_0^*]e^{-\lambda\eta_1\tau-\int_0^\tau\mu(\xi)d\xi}d\tau\\
&= q_0^*e^{\lambda\eta_1x+\int_0^x\mu(\tau)d\tau}\int_x^\infty(\lambda\eta_1+\mu(\tau))e^{-\lambda\eta_1-\int_0^\tau\mu(\xi)d\xi}d\tau\\
&= q_0^*e^{\lambda\eta_1x+\int_0^x\mu(\tau)d\tau}\left(-e^{-\lambda\eta_1\tau-\int_0^\tau\mu(\xi)d\xi}\Big|_x^\infty\right)\\
&= q_0^*,\quad \forall x\in[0,\infty).
\end{aligned}\tag{3-107}$$

Summarizing (3-98), (3-106) and (3-107) gives

$$|||q^*|||=\max\{|q_0^*|,\|q_1^*\|_{L^\infty[0,\infty)},\|q_2^*\|_{L^\infty[0,\infty)}\}=|q_0^*|<\infty.$$

From which together with Definition 1.22 we know that 0 is an eigenvalue of $(A+U+E)^*$. Moreover, by (3-106) and (3-107) it is easy to see that the eigenvector corresponding to 0 spans a one-dimensional linear space, that is, the geometric multiplicity of 0 is one (see Definition 1.23). □

Since Lemma 3.10 and Lemma 3.12 give $\langle p,q^*\rangle\neq0$, algebraic multiplicity of 0 is also 1 (for a detailed proof see Gupur et al. [65]).

Combining Lemma 3.10 and Theorem 3.1 with Definition 1.82 we conclude that the spectral bound of $A+U+E$ is zero, that is, $s(A+U+E)=0$. Thus by using Theorem 3.1, Corollary 3.9, Lemma 3.10 and Theorem 1.90 we conclude:

Theorem 3.13. *If $\mu(x)$ is Lipschitz continuous and satisfies $0<\underline{\mu}\leq\mu(x)\leq\overline{\mu}<\infty$, then there exist a positive projection $\mathbb{P}$ of rank 1 and suitable constants $\delta>0$ and $\mathcal{M}\geq 1$ such that*

$$\|T(t)-\mathbb{P}\|\leq\mathcal{M}e^{-\delta t},$$

where $\mathbb{P}=\frac{1}{2\pi i}\int_{\overline{\Gamma}}(zI-A-U-E)^{-1}dz$, $\overline{\Gamma}$ is a circle with center 0 and sufficiently small radius.

From the proof of Theorem 1.90, Theorem 3.1, Corollary 3.9 and Lemma 3.10 we know that

$$\{\gamma\in\sigma(A+U+E)\mid\Re\gamma=0\ \}=\{0\}.$$

In other words, all points on the imaginary axis except zero belong to the resolvent set of $A+U+E$. Thus by using Theorem 1.96, Lemma 3.10 and Lemma 3.12 we obtain the following result:

Theorem 3.14. *If $\mu(x)$ is Lipschitz continuous and satisfies $0<\overline{\mu}\leq\mu(x)\leq\overline{\mu}<\infty$, then the time-dependent solution of the system (3-15) strongly converges to its steady-state solution, that is,*

$$\lim_{t\to\infty}p(x,t)=\alpha p(x),$$

where $p(x)$ is the eigenvector corresponding to 0 (see Lemma 3.10).

In the following, when $\mu(x)=\mu$ by studying expression of $\mathbb{P}$ in Theorem 3.13 we will deduce that the time-dependent solution of the system (3-15) converges exponentially to its steady-state solution.

According to its definition, $\mathbb{P}$ is decided by its contour integral of the resolvent of $A+U+E$. In order to calculate the contour integral we should study the spectrum of $A+U+E$. Because Theorem 3.1, Corollary 3.9, Lemma 3.10 and Lemma 3.12 give descriptions of the spectrum of $A+U+E$ on the right half complex plane and on the imaginary axis, we need to research the spectrum of $A+U+E$ on the left half complex plane.

Lemma 3.15.

(1) *If $\lambda\eta_1<\mu\eta_2$, then $\sqrt{\lambda\eta_1\mu\eta_2}-\lambda\eta_1-\mu\eta_2$ is an eigenvalue of $A+U+E$ with geometric multiplicity 1. Moreover, there are no other eigenvalues of $A+U+E$ except zero in the circle $|z|=|\sqrt{\lambda\eta_1\mu\eta_2}-\lambda\eta_1-\mu\eta_2|$.*

(2) *If $\sqrt{\lambda\eta_1\mu\eta_2}+\lambda\eta_1<\mu q_2$, then $-\sqrt{\lambda\eta_1\mu\eta_2}-\lambda\eta_1-\mu\eta_2$ is an eigenvalue of $A+U+E$ with geometric multiplicity 1.*

Proof. Consider the equation $(\gamma I - A - U - E)p = 0$, i.e.,

$$(\gamma + \lambda\eta_1)p_0 = \mu\eta_2 \int_0^\infty p_1(x)dx, \tag{3-108}$$

$$\frac{dp_1(x)}{dx} = -(\gamma + \lambda\eta_1 + \mu)p_1(x), \tag{3-109}$$

$$\frac{dp_2(x)}{dx} = -(\gamma + \mu)p_2(x) + \lambda\eta_1 p_1(x), \tag{3-110}$$

$$\begin{aligned} p_1(0) = \lambda\eta_1 p_0 + \mu q_2 \int_0^\infty p_1(x)dx \\ + \mu\eta_2 \int_0^\infty p_2(x)dx, \end{aligned} \tag{3-111}$$

$$p_2(0) = \mu q_2 \int_0^\infty p_2(x)dx. \tag{3-112}$$

By solving (3-109) and (3-110) we have

$$p_1(x) = a_1 e^{-(\gamma+\lambda\eta_1+\mu)x}, \tag{3-113}$$

$$p_2(x) = a_2 e^{-(\gamma+\mu)x} + \lambda\eta_1 e^{-(\gamma+\mu)x} \int_0^x p_1(\tau)e^{(\gamma+\mu)\tau} d\tau. \tag{3-114}$$

Through inserting (3-113) into (3-108) we get (assume $\Re\gamma + \lambda\eta_1 + \mu > 0$)

$$\begin{aligned} (\gamma + \lambda\eta_1)p_0 = \mu\eta_2 \int_0^\infty a_1 e^{-(\gamma+\lambda\eta_1+\mu)x} dx \\ \Rightarrow a_1 = \frac{(\gamma + \lambda\eta_1)(\gamma + \lambda\eta_1 + \mu)}{\mu\eta_2} p_0. \end{aligned} \tag{3-115}$$

By combining (3-114) with (3-112) and using the Fubini theorem (see Theorem 1.49), (3-113), (3-115) and $\eta_2 + q_2 = 1$ we deduce (assume $\Re\gamma + \mu > 0$)

$$\begin{aligned} a_2 = p_2(0) = \mu q_2 \int_0^\infty a_2 e^{-(\gamma+\mu)x} dx \\ + \lambda\eta_1\mu q_2 \int_0^\infty e^{-(\gamma+\mu)x} \int_0^x p_1(\tau)e^{(\gamma+\mu)\tau} d\tau dx \\ = \frac{\mu q_2}{\gamma + \mu} a_2 + \lambda\eta_1\mu q_2 \int_0^\infty p_1(\tau)e^{(\gamma+\mu)\tau} \int_\tau^\infty e^{-(\gamma+\mu)x} dx d\tau \\ = \frac{\mu q_2}{\gamma + \mu} a_2 + \frac{\lambda\eta_1\mu q_2}{\gamma + \mu} \int_0^\infty p_1(\tau)d\tau \\ = \frac{\mu q_2}{\gamma + \mu} a_2 + \frac{\lambda\eta_1\mu q_2}{\gamma + \mu} \frac{a_1}{\gamma + \lambda\eta_1 + \mu} \\ \Longrightarrow \end{aligned}$$

$$\begin{aligned} a_2 &= \frac{\lambda\eta_1\mu q_2}{(\gamma+\mu\eta_2)(\gamma+\lambda\eta_1+\mu)}a_1 \\ &= \frac{\lambda\eta_1 q_2(\gamma+\lambda\eta_1)}{\eta_2(\gamma+\mu\eta_2)}p_0. \end{aligned} \tag{3-116}$$

Through inserting (3-113) and (3-114) into (3-111) and using $a_2 = p_2(0) = \mu q_2 \int_0^\infty p_2(x)dx$ and (3-116), we calculate

$$\begin{aligned} a_1 &= \lambda\eta_1 p_0 + \mu q_2 \int_0^\infty a_1 e^{-(\gamma+\lambda\eta_1+\mu)x}dx + \mu\eta_2 \int_0^\infty p_2(x)dx \\ &= \lambda\eta_1 p_0 + \frac{\mu q_2}{\gamma+\lambda\eta_1+\mu}a_1 + \frac{\eta_2}{q_2}a_2 \\ &= \lambda\eta_1 p_0 + \frac{\mu q_2}{\gamma+\lambda\eta_1+\mu}a_1 + \frac{\lambda\eta_1\mu\eta_2}{(\gamma+\mu\eta_2)(\gamma+\lambda\eta_1+\mu)}a_1 \\ &\Longrightarrow \\ &\left[1 - \frac{\mu q_2}{\gamma+\lambda\eta_1+\mu} - \frac{\lambda\eta_1\mu\eta_2}{(\gamma+\mu\eta_2)(\gamma+\lambda\eta_1+\mu)}\right]a_1 = \lambda\eta_1 p_0 \\ &\Longrightarrow \\ &\frac{(\gamma+\mu\eta_2)(\gamma+\lambda\eta_1+\mu) - \mu q_2(\gamma+\mu\eta_2) - \lambda\eta_1\mu\eta_2}{(\gamma+\mu\eta_2)(\gamma+\lambda\eta_1+\mu)}a_1 = \lambda\eta_1 p_0 \\ &\Longrightarrow \\ &\frac{(\gamma+\mu\eta_2)(\gamma+\mu\eta_2+\lambda\eta_1) - \lambda\eta_1\mu\eta_2}{(\gamma+\mu\eta_2)(\gamma+\lambda\eta_1+\mu)}a_1 = \lambda\eta_1 p_0 \\ &\Longrightarrow \\ &\frac{(\gamma+\mu\eta_2)^2 + \gamma\lambda\eta_1}{(\gamma+\mu\eta_2)(\gamma+\lambda\eta_1+\mu)}a_1 = \lambda\eta_1 p_0 \\ &\Longrightarrow \\ a_1 &= \frac{\lambda\eta_1(\gamma+\mu\eta_2)(\gamma+\lambda\eta_1+\mu)}{(\gamma+\mu\eta_2)^2+\gamma\lambda\eta_1}p_0. \end{aligned} \tag{3-117}$$

From (3-115) and (3-117) we derive, noting $p_0 \neq 0$,

$$\begin{aligned} &\frac{\lambda\eta_1(\gamma+\mu\eta_2)(\gamma+\lambda\eta_1+\mu)}{(\gamma+\mu\eta_2)^2+\gamma\lambda\eta_1}p_0 = \frac{(\gamma+\lambda\eta_1)(\gamma+\lambda\eta_1+\mu)}{\mu\eta_2}p_0 \\ &\Longrightarrow \\ &\lambda\eta_1\mu\eta_2(\gamma+\mu\eta_2) = (\gamma+\lambda\eta_1)\left[(\gamma+\mu\eta_2)^2+\gamma\lambda\eta_1\right] \\ &\Longrightarrow \\ &\gamma\lambda\eta_1\mu\eta_2 + \lambda\eta_1(\mu\eta_2)^2 \\ &= (\gamma+\lambda\eta_1)\left[\gamma^2 + 2\gamma\mu\eta_2 + (\mu\eta_2)^2 + \gamma\lambda\eta_1\right] \\ &\Longrightarrow \end{aligned}$$

$$\begin{aligned}
&\gamma\lambda\eta_1\mu\eta_2 + \lambda\eta_1(\mu\eta_2)^2 \\
&= \gamma\left[\gamma^2 + 2\gamma\mu\eta_2 + (\mu\eta_2)^2 + \gamma\lambda\eta_1\right] \\
&\quad + \lambda\eta_1\left[\gamma^2 + 2\gamma\mu\eta_2 + \gamma\lambda\eta_1\right] + \lambda\eta_1(\mu\eta_2)^2 \\
&\Longrightarrow \\
&\gamma\left[\gamma^2 + 2\gamma\lambda\eta_1 + 2\gamma\mu\eta_2 + (\lambda\eta_1)^2 + (\mu\eta_2)^2 + \lambda\eta_1\mu\eta_2\right] = 0 \\
&\Longrightarrow \\
&\gamma\left[\gamma^2 + \gamma(2\lambda\eta_1 + 2\mu\eta_2) + (\lambda\eta_1)^2 + (\mu\eta_2)^2 + \lambda\eta_1\mu\eta_2\right] = 0 \\
&\Longrightarrow \\
&\gamma = 0, &&(3\text{-}118)\\
&\gamma = \sqrt{\lambda\eta_1\mu\eta_2} - \lambda\eta_1 - \mu\eta_2, &&(3\text{-}119)\\
&\gamma = -\sqrt{\lambda\eta_1\mu\eta_2} - \lambda\eta_1 - \mu\eta_2. &&(3\text{-}120)
\end{aligned}$$

Since $\gamma = 0$ is the result in Lemma 3.10, we will discuss only (3-119) and (3-120). From the condition $\lambda\eta_1 < \mu\eta_2$ we know that $-\mu < -\mu\eta_2 < \sqrt{\lambda\eta_1\mu\eta_2} - \lambda\eta_1 - \mu\eta_2 < -\lambda\eta_1$ and therefore by inserting (3-119) into (3-115) and (3-116) we obtain

$$\begin{aligned}
a_1 &= \frac{(\gamma + \lambda\eta_1)(\gamma + \lambda\eta_1 + \mu)}{\mu\eta_2} p_0 \\
&= \frac{(\sqrt{\lambda\eta_1\mu\eta_2} - \mu\eta_2)(\sqrt{\lambda\eta_1\mu\eta_2} + \mu q_2)}{\mu\eta_2} p_0. &&(3\text{-}121)\\
a_2 &= \frac{\lambda\eta_1 q_2(\gamma + \lambda\eta_1)}{\eta_2(\gamma + \mu\eta_2)} p_0 \\
&= \frac{\lambda\eta_1 q_2(\sqrt{\lambda\eta_1\mu\eta_2} - \mu\eta_2)}{\eta_2(\sqrt{\lambda\eta_1\mu\eta_2} - \lambda\eta_1)} p_0. &&(3\text{-}122)
\end{aligned}$$

By summarizing (3-113), (3-114), (3-115), (3-116), (3-119), (3-121) and (3-122) and applying the Fubini theorem (see Theorem 1.49) we deduce

$$\begin{aligned}
\|p\| &= |p_0| + \|p_1\|_{L^1[0,\infty)} + \|p_2\|_{L^1[0,\infty)} \\
&= |p_0| + |a_1| \int_0^\infty \left|e^{-(\gamma+\lambda\eta_1+\mu)x}\right| dx \\
&\quad + \int_0^\infty \left| a_2 e^{-(\gamma+\mu)x} + \lambda\eta_1 e^{-(\gamma+\mu)x} \int_0^x p_1(\tau) e^{(\gamma+\mu)\tau} d\tau \right| dx \\
&\le |p_0| + \frac{|a_1|}{\gamma + \lambda\eta_1 + \mu} + \int_0^\infty \left| a_2 e^{-(\gamma+\mu)x}\right| dx \\
&\quad + \int_0^\infty \lambda\eta_1 \left| e^{-(\gamma+\mu)x} \int_0^x p_1(\tau) e^{(\gamma+\mu)\tau} d\tau \right| dx \\
&\le |p_0| + \frac{|a_1|}{\gamma + \lambda\eta_1 + \mu} + |a_2| \int_0^\infty \left| e^{-(\gamma+\mu)x}\right| dx
\end{aligned}$$

$$
\begin{aligned}
&+\int_0^\infty \lambda\eta_1 \left|e^{(\gamma+\mu)\tau}p_1(\tau)\right| \int_\tau^\infty \left|e^{-(\gamma+\mu)x}\right| dx d\tau \\
&\le |p_0| + \frac{|a_1|}{\gamma+\lambda\eta_1+\mu} + \frac{|a_2|}{\gamma+\mu} + \frac{\lambda\eta_1}{\gamma+\mu}\int_0^\infty |p_1(\tau)|\, d\tau \\
&= |p_0| + \frac{\left|\sqrt{\lambda\eta_1\mu\eta_2}-\mu\eta_2\right|}{\mu\eta_2}|p_0| \\
&\quad + \frac{\lambda\eta_1 q_2|\sqrt{\lambda\eta_1\mu\eta_2}-\mu\eta_2|}{\eta_2(\sqrt{\lambda\eta_1\mu\eta_2}-\lambda\eta_1)(\sqrt{\lambda\eta_1\mu\eta_2}-\lambda\eta_1+\mu q_2)}|p_0| \\
&\quad + \frac{\lambda\eta_1\left|\sqrt{\lambda\eta_1\mu\eta_2}-\mu\eta_2\right|}{\mu\eta_2(\sqrt{\lambda\eta_1\mu\eta_2}-\lambda\eta_1+\mu q_2)}|p_0| \\
&< \infty.
\end{aligned}
$$

Which shows that $\sqrt{\lambda\eta_1\mu\eta_2}-\lambda\eta_1-\mu\eta_2$ is an eigenvalue of $A+U+E$. Moreover, from (3-113), (3-114), (3-121), (3-122) we know that the eigenvector space corresponding to $\sqrt{\lambda\eta_1\mu\eta_2}-\lambda\eta_1-\mu\eta_2$ is a one-dimensional linear space, that is, the geometric multiplicity of $\sqrt{\lambda\eta_1\mu\eta_2}-\lambda\eta_1-\mu\eta_2$ is 1 (see Definition 1.23). From the above analysis we see that there are no other eigenvalues of $A+U+E$ except zero in the circle $|z| = |\sqrt{\lambda\eta_1\mu\eta_2}-\lambda\eta_1-\mu\eta_2|$.

The condition $\sqrt{\lambda\eta_1\mu\eta_2}+\lambda\eta_1 < \mu q_2$ implies $-\sqrt{\lambda\eta_1\mu\eta_2}-\lambda\eta_1-\mu\eta_2 > -\mu$. Hence by combining (3-113), (3-114), (3-115), (3-116) with (3-120) and using the Fubini theorem (see Theorem 1.49) we estimate

$$
\begin{aligned}
\|p\| &= |p_0| + \|p_1\|_{L^1[0,\infty)} + \|p_2\|_{L^1[0,\infty)} \\
&= |p_0| + |a_1|\int_0^\infty \left|e^{-(\gamma+\mu+\lambda\eta_1)x}\right| dx \\
&\quad + \int_0^\infty \left|a_2 e^{-(\gamma+\mu)x} + \lambda\eta_1 e^{-(\gamma+\mu)x}\int_0^x p_1(\tau)e^{(\gamma+\mu)\tau}d\tau\right| dx \\
&\le |p_0| + \frac{|a_1|}{\gamma+\mu+\lambda\eta_1} + \int_0^\infty \left|a_2 e^{-(\gamma+\mu)x}\right| dx \\
&\quad + \int_0^\infty \lambda\eta_1 \left|e^{-(\gamma+\mu)x}\int_0^x p_1(\tau)e^{(\gamma+\mu)\tau}d\tau\right| dx \\
&\le |p_0| + \frac{|a_1|}{\gamma+\mu+\lambda\eta_1} + |a_2|\int_0^\infty \left|e^{-(\gamma+\mu)x}\right| dx \\
&\quad + \int_0^\infty \lambda\eta_1 \left|e^{(\gamma+\mu)\tau}p_1(\tau)\right| \int_\tau^\infty \left|e^{-(\gamma+\mu)x}\right| dx d\tau \\
&\le |p_0| + \frac{|a_1|}{\gamma+\mu+\lambda\eta_1} + \frac{|a_2|}{\gamma+\mu} + \frac{\lambda\eta_1}{\gamma+\mu}\int_0^\infty |p_1(\tau)| d\tau \\
&= |p_0| + \frac{|\gamma+\lambda\eta_1||\gamma+\mu+\lambda\eta_1|}{\mu\eta_2(\gamma+\mu+\lambda\eta_1)}|p_0|
\end{aligned}
$$

$$
\begin{aligned}
&+ \frac{\lambda\eta_1 q_2|\gamma + \lambda\eta_1|}{\eta_2|\gamma + \mu\eta_2|(\gamma + \mu)}|p_0| \\
&+ \frac{\lambda\eta_1|\gamma + \lambda\eta_1||\gamma + \mu + \lambda\eta_1|}{\mu\eta_2(\gamma + \mu + \lambda\eta_1)(\gamma + \mu)}|p_0| \\
= {}& |p_0| + \frac{\sqrt{\lambda\eta_1\mu\eta_2} + \mu\eta_2}{\mu\eta_2}|p_0| \\
&+ \frac{\lambda\eta_1 q_2(\sqrt{\lambda\eta_1\mu\eta_2} + \mu\eta_2)}{\eta_2(\sqrt{\lambda\eta_1\mu\eta_2} + \lambda\eta_1)\left|-\sqrt{\lambda\eta_1\mu\eta_2} - \lambda\eta_1 + \mu q_2\right|}|p_0| \\
&+ \frac{\lambda\eta_1(\sqrt{\lambda\eta_1\mu\eta_2} + \mu\eta_2)}{\mu\eta_2\left|-\sqrt{\lambda\eta_1\mu\eta_2} - \lambda\eta_1 + \mu q_2\right|}|p_0| \\
< {}& \infty.
\end{aligned}
$$

Which shows that $-\sqrt{\lambda\eta_1\mu\eta_2} - \lambda\eta_1 - \mu\eta_2$ is an eigenvalue of $A+U+E$. In addition, from (3-113), (3-114), (3-121), (3-122) it is not difficult to see that the eigenvectors corresponding to $-\sqrt{\lambda\eta_1\mu\eta_2} - \lambda\eta_1 - \mu\eta_2$ span a one-dimensional linear space, that is, the geometric multiplicity of $-\sqrt{\lambda\eta_1\mu\eta_2} - \lambda\eta_1 - \mu\eta_2$ is 1 (see Definition 1.23). □

Lemma 3.16.

(1) *When* $\lambda\eta_1 < \mu\eta_2$, $\sqrt{\lambda\eta_1\mu\eta_2} - \lambda\eta_1 - \mu\eta_2$ *is an eigenvalue of* $(A + U + E)^*$ *with geometric multiplicity* 1. *Moreover, there are no other eigenvalues of* $(A + U + E)^*$ *except zero in the circle* $|z| = |\sqrt{\lambda\eta_1\mu\eta_2} - \lambda\eta_1 - \mu\eta_2|$.

(2) *When* $\sqrt{\lambda\eta_1\mu\eta_2} + \lambda\eta_1 < \mu q_2$, $-\sqrt{\lambda\eta_1\mu\eta_2} - \lambda\eta_1 - \mu\eta_2$ *is an eigenvalue of* $(A + U + E)^*$ *with geometric multiplicity* 1.

Proof. Consider the equation $[\gamma I - (A + U + E)^*]q^* = 0$. By Lemma 3.11 it is equivalent to

$$
\begin{aligned}
&(\gamma + \lambda\eta_1)q_0^* = \lambda\eta_1 q_1^*(0), && \text{(3-123)}\\
&\frac{dq_1^*(x)}{dx} = (\gamma + \lambda\eta_1 + \mu)q_1^*(x) \\
&\qquad\qquad - \mu\eta_2 q_0^* - \mu q_2 q_1^*(0) - \lambda\eta_1 q_2^*(x), && \text{(3-124)}\\
&\frac{dq_2^*(x)}{dx} = (\gamma + \mu)q_2^*(x) - \mu\eta_2 q_1^*(0) - \mu q_2 q_2^*(0), && \text{(3-125)}\\
&q_1^*(\infty) = q_2^*(\infty) = \alpha. && \text{(3-126)}
\end{aligned}
$$

By solving (3-123)–(3-125) we have (assume $\Re\gamma + \mu + \lambda\eta_1 > 0$)

$$
\begin{aligned}
&q_1^*(0) = \frac{\gamma + \lambda\eta_1}{\lambda\eta_1}q_0^*, && \text{(3-127)}\\
&q_1^*(x) = b_1 e^{(\gamma+\lambda\eta_1+\mu)x} - e^{(\gamma+\lambda\eta_1+\mu)x}\int_0^x [\mu\eta_2 q_0^*
\end{aligned}
$$

$$
\begin{aligned}
&+ \mu q_2 q_1^*(0) + \lambda\eta_1 q_2^*(\tau)]e^{-(\gamma+\lambda\eta_1+\mu)\tau} d\tau \\
=&b_1 e^{(\gamma+\lambda\eta_1+\mu)x} - \lambda\eta_1 e^{(\gamma+\lambda\eta_1+\mu)x} \int_0^x q_2^*(\tau) e^{-(\gamma+\lambda\eta_1+\mu)\tau} d\tau \\
&+ \left[1 - e^{(\gamma+\lambda\eta_1+\mu)x}\right] \frac{\mu\eta_2 q_0^* + \mu q_2 q_1^*(0)}{\gamma+\lambda\eta_1+\mu},
\end{aligned} \tag{3-128}
$$

$$
\begin{aligned}
q_2^*(x) =& b_2 e^{(\gamma+\mu)x} \\
&- e^{(\gamma+\mu)x} \int_0^x \left[\mu\eta_2 q_1^*(0) + \mu q_2 q_2^*(0)\right] e^{-(\gamma+\mu)\tau} d\tau \\
=& b_2 e^{(\gamma+\mu)x} + \left[1 - e^{(\gamma+\mu)x}\right] \frac{\mu\eta_2 q_1^*(0) + \mu q_2 q_2^*(0)}{\gamma+\mu}.
\end{aligned} \tag{3-129}
$$

Through multiplying $e^{-(\gamma+\mu+\lambda\eta_1)x}$ to two sides of (3-128) and $e^{-(\gamma+\mu)x}$ to two sides of (3-129), and then taking the limit $x \to \infty$ and using (3-126) we obtain (assume $\Re\gamma + \mu > 0$)

$$
\begin{aligned}
b_1 = \lambda\eta_1 &\int_0^\infty q_2^*(\tau) e^{-(\gamma+\lambda\eta_1+\mu)\tau} d\tau \\
&+ \frac{\mu\eta_2 q_0^* + \mu q_2 q_1^*(0)}{\gamma+\lambda\eta_1+\mu},
\end{aligned} \tag{3-130}
$$

$$
b_2 = \frac{\mu\eta_2 q_1^*(0) + \mu q_2 q_2^*(0)}{\gamma+\mu}. \tag{3-131}
$$

By inserting (3-130) into (3-128) and (3-131) into (3-129) it gives

$$
\begin{aligned}
q_1^*(x) =& \lambda\eta_1 e^{(\gamma+\lambda\eta_1+\mu)x} \int_0^\infty q_2^*(\tau) e^{-(\gamma+\lambda\eta_1+\mu)\tau} d\tau \\
&+ \frac{\mu\eta_2 q_0^* + \mu q_2 q_1^*(0)}{\gamma+\lambda\eta_1+\mu} e^{(\gamma+\lambda\eta_1+\mu)x} \\
&- \lambda\eta_1 e^{(\gamma+\lambda\eta_1+\mu)x} \int_0^x q_2^*(\tau) e^{-(\gamma+\lambda\eta_1+\mu)\tau} d\tau \\
&+ \left[1 - e^{(\gamma+\lambda\eta_1+\mu)x}\right] \frac{\mu\eta_2 q_0^* + \mu q_2 q_1^*(0)}{\gamma+\lambda\eta_1+\mu} \\
=& \lambda\eta_1 e^{(\gamma+\lambda\eta_1+\mu)x} \int_x^\infty q_2^*(\tau) e^{-(\gamma+\lambda\eta_1+\mu)\tau} d\tau \\
&+ \frac{\mu\eta_2 q_0^* + \mu q_2 q_1^*(0)}{\gamma+\lambda\eta_1+\mu},
\end{aligned} \tag{3-132}
$$

$$
\begin{aligned}
q_2^*(x) =& \frac{\mu\eta_2 q_1^*(0) + \mu q_2 q_2^*(0)}{\gamma+\mu} e^{(\gamma+\mu)x} + \frac{\mu\eta_2 q_1^*(0) + \mu q_2 q_2^*(0)}{\gamma+\mu} \\
&- \frac{\mu\eta_2 q_1^*(0) + \mu q_2 q_2^*(0)}{\gamma+\mu} e^{(\gamma+\mu)x} \\
=& \frac{\mu\eta_2 q_1^*(0) + \mu q_2 q_2^*(0)}{\gamma+\mu}.
\end{aligned} \tag{3-133}
$$

From (3-133), $\eta_2 + q_2 = 1$ and (3-127) we deduce

$$\begin{aligned}
q_2^*(0) &= \frac{\mu\eta_2 q_1^*(0) + \mu q_2 q_2^*(0)}{\gamma + \mu} \\
&\Longrightarrow \\
(\gamma + \mu) q_2^*(0) &= \mu\eta_2 q_1^*(0) + \mu q_2 q_2^*(0) \\
&\Longrightarrow \\
(\gamma + \mu - \mu q_2) q_2^*(0) &= \mu\eta_2 q_1^*(0) \\
&\Longrightarrow \\
(\gamma + \mu\eta_2) q_2^*(0) &= \mu\eta_2 q_1^*(0) \\
&\Longrightarrow \\
q_2^*(x) &= \frac{\mu\eta_2 q_1^*(0) + \mu q_2 q_2^*(0)}{\gamma + \mu} \\
&= \frac{(\gamma + \mu\eta_2) q_2^*(0) + \mu q_2 q_2^*(0)}{\gamma + \mu} \\
&= \frac{(\gamma + \mu\eta_2 + \mu q_2) q_2^*(0)}{\gamma + \mu} \\
&= q_2^*(0) = \frac{\mu\eta_2}{\gamma + \mu\eta_2} q_1^*(0) \\
&= \frac{\mu\eta_2(\gamma + \lambda\eta_1)}{\lambda\eta_1(\gamma + \mu\eta_2)} q_0^*. \qquad (3\text{-}134)
\end{aligned}$$

Through inserting (3-134) into (3-132) we derive

$$\begin{aligned}
q_1^*(x) &= \lambda\eta_1 e^{(\gamma + \lambda\eta_1 + \mu)x} \int_x^\infty q_2^*(0) e^{-(\gamma + \lambda\eta_1 + \mu)\tau} d\tau \\
&\quad + \frac{\mu\eta_2 q_0^* + \mu q_2 q_1^*(0)}{\gamma + \lambda\eta_1 + \mu} \\
&= \frac{\lambda\eta_1}{\gamma + \lambda\eta_1 + \mu} q_2^*(0) + \frac{\mu\eta_2 q_0^* + \mu q_2 q_1^*(0)}{\gamma + \lambda\eta_1 + \mu} \\
&\Longrightarrow \\
q_1^*(0) &= \frac{\lambda\eta_1 \mu\eta_2}{(\gamma + \mu\eta_2)(\gamma + \lambda\eta_1 + \mu)} q_1^*(0) + \frac{\mu\eta_2 q_0^* + \mu q_2 q_1^*(0)}{\gamma + \lambda\eta_1 + \mu} \\
&= \frac{\lambda\eta_1 \mu\eta_2}{(\gamma + \mu\eta_2)(\gamma + \lambda\eta_1 + \mu)} q_1^*(0) + \frac{\mu q_2}{\gamma + \lambda\eta_1 + \mu} q_1^*(0) \\
&\quad + \frac{\mu\eta_2}{\gamma + \lambda\eta_1 + \mu} q_0^* \\
&= \frac{\lambda\eta_1 \mu\eta_2 + \mu q_2(\gamma + \mu\eta_2)}{(\gamma + \mu\eta_2)(\gamma + \lambda\eta_1 + \mu)} q_1^*(0) \\
&\quad + \frac{\mu\eta_2}{\gamma + \lambda\eta_1 + \mu} q_0^*
\end{aligned}$$

$$
\begin{aligned}
&\Longrightarrow \\
&\frac{(\gamma+\mu\eta_2)^2+\gamma\lambda\eta_1}{(\gamma+\mu\eta_2)(\gamma+\lambda\eta_1+\mu)}q_1^*(0) = \frac{\mu\eta_2}{\gamma+\lambda\eta_1+\mu}q_0^* \\
&\Longrightarrow \\
q_1^*(x) &= \frac{\lambda\eta_1\mu\eta_2+\mu q_2(\gamma+\mu\eta_2)}{(\gamma+\mu\eta_2)(\gamma+\lambda\eta_1+\mu)}q_1^*(0) \\
&\quad + \frac{\mu\eta_2}{\gamma+\lambda\eta_1+\mu}q_0^* \\
&= q_1^*(0) = \frac{\mu\eta_2(\gamma+\mu\eta_2)}{(\gamma+\mu\eta_2)^2+\gamma\lambda\eta_1}q_0^*. \qquad (3\text{-}135)
\end{aligned}
$$

It is the same as (3-118), (3-119) and (3-120), by comparing (3-127) with (3-135) and noting $q_0^* \neq 0$ it follows that

$$
\begin{aligned}
&\frac{\mu\eta_2(\gamma+\mu\eta_2)}{\gamma^2+2\gamma\mu\eta_2+\gamma\lambda\eta_1+(\mu\eta_2)^2}q_0^* = \frac{\gamma+\lambda\eta_1}{\lambda\eta_1}q_0^* \\
&\Longrightarrow \\
&\lambda\eta_1\mu\eta_2(\gamma+\mu\eta_2) = (\gamma+\lambda\eta_1)\left[\gamma^2+2\gamma\mu\eta_2+\gamma\lambda\eta_1+(\mu\eta_2)^2\right] \\
&\Longrightarrow \\
&\gamma\lambda\eta_1\mu\eta_2+\lambda\eta_1(\mu\eta_2)^2 \\
&= (\gamma+\lambda\eta_1)\left[\gamma^2+2\gamma\mu\eta_2+(\mu\eta_2)^2+\gamma\lambda\eta_1\right] \\
&= \gamma\left[\gamma^2+2\gamma\mu\eta_2+(\mu\eta_2)^2+\gamma\lambda\eta_1\right] \\
&\quad + \lambda\eta_1\left[\gamma^2+\gamma\lambda\eta_1\right]+2\gamma\lambda\eta_1\mu\eta_2+\lambda\eta_1(\mu\eta_2)^2 \\
&\Longrightarrow \\
&\gamma\left[\gamma^2+2\gamma\lambda\eta_1+2\gamma\mu\eta_2+(\lambda\eta_1)^2+(\mu\eta_2)^2+\lambda\eta_1\mu\eta_2\right] = 0 \\
&\Longrightarrow \\
&\gamma\left[\gamma^2+\gamma(2\lambda\eta_1+2\mu\eta_2)+(\lambda\eta_1)^2+(\mu\eta_2)^2+\lambda\eta_1\mu\eta_2\right] = 0 \\
&\Longrightarrow \\
&\gamma = 0, \qquad (3\text{-}136) \\
&\gamma = \sqrt{\lambda\eta_1\mu\eta_2}-\lambda\eta_1-\mu\eta_2, \qquad (3\text{-}137) \\
&\gamma = -\sqrt{\lambda\eta_1\mu\eta_2}-\lambda\eta_1-\mu\eta_2. \qquad (3\text{-}138)
\end{aligned}
$$

$\gamma = 0$ is just the result in Lemma 3.12 and therefore, in the following we discuss only (3-137) and (3-138). The condition $\lambda\eta_1 < \mu\eta_2$ implies $-\mu < -\mu\eta_2 < \sqrt{\lambda\eta_1\mu\eta_2}-\lambda\eta_1-\mu\eta_2 < -\lambda\eta_1$, so by inserting (3-137) into (3-127) and (3-134) and using (3-135) we calculate

$$
q_1^*(x) = \frac{\sqrt{\lambda\eta_1\mu\eta_2}-\mu\eta_2}{\lambda\eta_1}q_0^*, \qquad (3\text{-}139)
$$

$$q_2^*(x) = \frac{\mu\eta_2(\sqrt{\lambda\eta_1\mu\eta_2} - \mu\eta_2)}{\lambda\eta_1(\sqrt{\lambda\eta_1\mu\eta_2} - \lambda\eta_1)} q_0^*. \tag{3-140}$$

(3-139) and (3-140) give

$$\begin{aligned}
|||q^*||| &= \max\left\{|q_0^*|, \|q_1^*\|_{L^\infty[0,\infty)}, \|q_2^*\|_{L^\infty[0,\infty)}\right\} \\
&= \max\left\{|q_0^*|, \left|\frac{\sqrt{\lambda\eta_1\mu\eta_2} - \mu\eta_2}{\lambda\eta_1}\right| |q_0^*|,\right. \\
&\qquad \left.\left|\frac{\mu\eta_2(\sqrt{\lambda\eta_1\mu\eta_2} - \mu\eta_2)}{\lambda\eta_1(\sqrt{\lambda\eta_1\mu\eta_2} - \lambda\eta_1)}\right| |q_0^*|\right\} \\
&< \infty.
\end{aligned}$$

Which shows that $\sqrt{\lambda\eta_1\mu\eta_2} - \lambda\eta_1 - \mu\eta_2$ is an eigenvalue of $(A+U+E)^*$ (see Definition 1.22). From the condition $\sqrt{\lambda\eta_1\mu\eta_2} + \lambda\eta_1 < \mu q_2$ we know that $-\sqrt{\lambda\eta_1\mu\eta_2} - \lambda\eta_1 - \mu\eta_2 > -\mu$, from which together with (3-127), (3-134) and (3-138) and using (3-135) we get

$$\begin{aligned}
|||q^*||| &= \max\left\{|q_0^*|, \|q_1^*\|_{L^\infty[0,\infty)}, \|q_2^*\|_{L^\infty[0,\infty)}\right\} \\
&= \max\left\{|q_0^*|, \left|\frac{-\sqrt{\lambda\eta_1\mu\eta_2} - \mu\eta_2}{\lambda\eta_1}\right| |q_0^*|,\right. \\
&\qquad \left.\left|\frac{\mu\eta_2(-\sqrt{\lambda\eta_1\mu\eta_2} - \mu\eta_2)}{\lambda\eta_1(-\sqrt{\lambda\eta_1\mu\eta_2} - \lambda\eta_1)}\right| |q_0^*|\right\} \\
&= \max\left\{|q_0^*|, \frac{\sqrt{\lambda\eta_1\mu\eta_2} + \mu\eta_2}{\lambda\eta_1} |q_0^*|,\right. \\
&\qquad \left.\frac{\mu\eta_2(\sqrt{\lambda\eta_1\mu\eta_2} + \mu\eta_2)}{\lambda\eta_1(\sqrt{\lambda\eta_1\mu\eta_2} + \lambda\eta_1)} |q_0^*|\right\} \\
&< \infty.
\end{aligned}$$

Which shows that $-\sqrt{\lambda\eta_1\mu\eta_2} - \lambda\eta_1 - \mu\eta_2$ is an eigenvalue of $(A + U + E)^*$ (see Definition 1.22). Moreover, by (3-134), (3-135) it is easy to see that the eigenvector space corresponding to $\pm\sqrt{\lambda\eta_1\mu\eta_2} - \lambda\eta_1 - \mu\eta_2$ is a one-dimensional linear space, i.e., the geometric multiplicity of $\pm\sqrt{\lambda\eta_1\mu\eta_2} - \lambda\eta_1 - \mu\eta_2$ is 1 (see Definition 1.23). From the above analysis we see that there are no other eigenvalues of $(A+U+E)^*$ except zero in the circle $|z| = |\sqrt{\lambda\eta_1\mu\eta_2} - \lambda\eta_1 - \mu\eta_2|$. □

Lemma 3.17. *For $z \in \rho(A + U + E)$ we have*

$$(zI - A - U - E)^{-1} y = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \end{pmatrix}, \quad \forall y \in X,$$

where

$$\psi_1 = \frac{1}{(z + \lambda\eta_1)} y_0 + \frac{\mu\eta_2}{(z + \lambda\eta_1)(z + \mu + \lambda\eta_1)} \int_0^\infty y_1(x)dx$$

$$
\begin{aligned}
&+ \frac{\lambda\eta_1\mu\eta_2(z+\mu\eta_2)}{z(z+\lambda\eta_1)[(z+\lambda\eta_1+\mu\eta_2)^2-\lambda\eta_1\mu\eta_2]} y_0 \\
&+ \left\{ \frac{\mu\eta_2\lambda\eta_1\mu\eta_2}{z(z+\mu+\lambda\eta_1)[(z+\lambda\eta_1+\mu\eta_2)^2-\lambda\eta_1\mu\eta_2]} \right. \\
&+ \left. \frac{\mu\eta_2(z\mu q_2+\lambda\eta_1\mu)(z+\mu\eta_2)}{z(z+\lambda\eta_1)(z+\mu+\lambda\eta_1)[(z+\lambda\eta_1+\mu\eta_2)^2-\lambda\eta_1\mu\eta_2]} \right\} \\
&\times \int_0^\infty y_1(x)dx \\
&+ \frac{(\mu\eta_2)^2}{z[(z+\lambda\eta_1+\mu\eta_2)^2-\lambda\eta_1\mu\eta_2]} \int_0^\infty y_2(x)dx,
\end{aligned}
$$

$$
\begin{aligned}
\psi_2 = {} & e^{-(z+\mu+\lambda\eta_1)x} \int_0^x e^{(z+\mu+\lambda\eta_1)\tau} y_1(\tau)d\tau \\
&+ \frac{\lambda\eta_1(z+\mu\eta_2)(z+\mu+\lambda\eta_1)}{z[(z+\lambda\eta_1+\mu\eta_2)^2-\lambda\eta_1\mu\eta_2]} e^{-(z+\mu+\lambda\eta_1)x} y_0 \\
&+ \frac{(\lambda\eta_1)^2\mu\eta_2+(z+\mu\eta_2)(z\mu q_2+\lambda\eta_1\mu)}{z[(z+\lambda\eta_1+\mu\eta_2)^2-\lambda\eta_1\mu\eta_2]} \\
&\times e^{-(z+\mu+\lambda\eta_1)x} \int_0^\infty y_1(x)dx \\
&+ \frac{\mu\eta_2(z+\lambda\eta_1)(z+\mu+\lambda\eta_1)}{z[(z+\lambda\eta_1+\mu\eta_2)^2-\lambda\eta_1\mu\eta_2]} e^{-(z+\mu+\lambda\eta_1)x} \int_0^\infty y_2(x)dx,
\end{aligned}
$$

$$
\begin{aligned}
\psi_3 = {} & e^{-(z+\mu)x} \int_0^x e^{(z+\mu)\xi} y_1(\xi)d\xi \\
&- e^{-(z+\mu+\lambda\eta_1)x} \int_0^x e^{(z+\mu+\lambda\eta_1)\xi} y_1(\xi)d\xi \\
&+ e^{-(z+\mu)x} \int_0^x e^{(z+\mu)\tau} y_2(\tau)d\tau \\
&+ \frac{(\lambda\eta_1)^2\mu q_2+\lambda\eta_1(z+\mu\eta_2)(z+\mu+\lambda\eta_1)}{z[(z+\lambda\eta_1+\mu\eta_2)^2-\lambda\eta_1\mu\eta_2]} e^{-(z+\mu)x} y_0 \\
&- \frac{\lambda\eta_1(z+\mu\eta_2)(z+\mu+\lambda\eta_1)}{z[(z+\lambda\eta_1+\mu\eta_2)^2-\lambda\eta_1\mu\eta_2]} e^{-(z+\mu+\lambda\eta_1)x} y_0 \\
&+ \left\{ \frac{\lambda\eta_1\mu q_2(z+\lambda\eta_1)+\lambda\eta_1\mu\eta_2(z+\lambda\eta_1)}{z[(z+\lambda\eta_1+\mu\eta_2)^2-\lambda\eta_1\mu\eta_2]} \right. \\
&+ \left. \frac{(z\mu q_2+\lambda\eta_1\mu)(z+\mu\eta_2)}{z[(z+\lambda\eta_1+\mu\eta_2)^2-\lambda\eta_1\mu\eta_2]} \right\} e^{-(z+\mu)x} \int_0^\infty y_1(x)dx \\
&- \frac{[\lambda\eta_1\mu\eta_2(z+\lambda\eta_1)+(z+\mu\eta_2)(z\mu q_2+\lambda\eta_1\mu)]}{z[(z+\lambda\eta_1+\mu\eta_2)^2-\lambda\eta_1\mu\eta_2]}
\end{aligned}
$$

$$
\begin{aligned}
&\times e^{-(z+\mu+\lambda\eta_1)x}\int_0^\infty y_1(x)dx\\
&+\left\{\frac{\mu q_2[(z+\lambda\eta_1)^2+z\mu\eta_2]}{z[(z+\lambda\eta_1+\mu\eta_2)^2-\lambda\eta_1\mu\eta_2]}\right.\\
&+\left.\frac{\mu\eta_2(z+\lambda\eta_1)(z+\mu+\lambda\eta_1)}{z[(z+\lambda\eta_1+\mu\eta_2)^2-\lambda\eta_1\mu\eta_2]}\right\}\\
&\quad\times e^{-(z+\mu)x}\int_0^\infty y_2(x)dx\\
&-\frac{\mu\eta_2(z+\lambda\eta_1)(z+\mu+\lambda\eta_1)}{z[(z+\lambda\eta_1+\mu\eta_2)^2-\lambda\eta_1\mu\eta_2]}\\
&\quad\times e^{-(z+\mu+\lambda\eta_1)x}\int_0^\infty y_2(x)dx.
\end{aligned}
$$

Proof. For any given $y \in X$ consider the equation $(zI - A - U - E)p = y$, that is,

$$(z+\lambda\eta_1)p_0 = \mu\eta_2\int_0^\infty p_1(x)dx + y_0, \tag{3-141}$$

$$\frac{dp_1(x)}{dx} = -(z+\mu+\lambda\eta_1)p_1(x) + y_1(x), \tag{3-142}$$

$$\frac{dp_2(x)}{dx} = -(z+\mu)p_2(x) + \lambda\eta_1 p_1(x) + y_2(x), \tag{3-143}$$

$$
\begin{aligned}
p_1(0) = \lambda\eta_1 p_0 + \mu q_2\int_0^\infty p_1(x)dx&\\
+\mu\eta_2\int_0^\infty p_2(x)dx&,
\end{aligned}\tag{3-144}
$$

$$p_2(0) = \mu q_2\int_0^\infty p_2(x)dx. \tag{3-145}$$

Solving (3-141)–(3-143) we have

$$p_0 = \frac{y_0 + \mu\eta_2\int_0^\infty p_1(x)dx}{z+\lambda\eta_1}, \tag{3-146}$$

$$
\begin{aligned}
p_1(x) = {}& p_1(0)e^{-(z+\mu+\lambda\eta_1)x}\\
&+ e^{-(z+\mu+\lambda\eta_1)x}\int_0^x e^{(z+\mu+\lambda\eta_1)\tau}y_1(\tau)d\tau,
\end{aligned}\tag{3-147}
$$

$$
\begin{aligned}
p_2(x) = {}& p_2(0)e^{-(z+\mu)x} + \lambda\eta_1 e^{-(z+\mu)x}\int_0^x e^{(z+\mu)\tau}p_1(\tau)d\tau\\
&+ e^{-(z+\mu)x}\int_0^x e^{(z+\mu)\tau}y_2(\tau)d\tau.
\end{aligned}\tag{3-148}
$$

By combining (3-146), (3-147) and (3-148) with the Fubini theorem (see Theorem 1.49) it follows that (assume $\Re z + \mu + \lambda\eta_1 > 0$)

$$p_0 = \frac{y_0}{z+\lambda\eta_1} + \frac{\mu\eta_2\left[p_1(0) + \int_0^\infty y_1(x)dx\right]}{(z+\mu+\lambda\eta_1)(z+\lambda\eta_1)}, \tag{3-149}$$

$$\begin{aligned} p_1(x) = {} & p_1(0)e^{-(z+\mu+\lambda\eta_1)x} \\ & + e^{-(z+\mu+\lambda\eta_1)x}\int_0^x e^{(z+\mu+\lambda\eta_1)\tau} y_1(\tau)d\tau, \end{aligned} \tag{3-150}$$

$$\begin{aligned} p_2(x) = {} & p_2(0)e^{-(z+\mu)x} + p_1(0)\left[e^{-(z+\mu)x} - e^{-(z+\mu+\lambda\eta_1)x}\right] \\ & + \lambda\eta_1 e^{-(z+\mu)x}\int_0^x e^{-\lambda\eta_1\tau}\int_0^\tau e^{(z+\mu+\lambda\eta_1)\xi} y_1(\xi)d\xi d\tau \\ & + e^{-(z+\mu)x}\int_0^x e^{(z+\mu)\tau} y_2(\tau)d\tau \\ = {} & p_2(0)e^{-(z+\mu)x} + p_1(0)\left[e^{-(z+\mu)x} - e^{-(z+\mu+\lambda\eta_1)x}\right] \\ & + \lambda\eta_1 e^{-(z+\mu)x}\int_0^x e^{(z+\mu+\lambda\eta_1)\xi} y_1(\xi)\int_\xi^x e^{-\lambda\eta_1\tau}d\tau d\xi \\ & + e^{-(z+\mu)x}\int_0^x e^{(z+\mu)\tau} y_2(\tau)d\tau \\ = {} & p_2(0)e^{-(z+\mu)x} + p_1(0)\left[e^{-(z+\mu)x} - e^{-(z+\mu+\lambda\eta_1)x}\right] \\ & + e^{-(z+\mu)x}\int_0^x e^{(z+\mu)\xi} y_1(\xi)d\xi \\ & - e^{-(z+\mu+\lambda\eta_1)x}\int_0^x e^{(z+\mu+\lambda\eta_1)\xi} y_1(\xi)d\xi \\ & + e^{-(z+\mu)x}\int_0^x e^{(z+\mu)\tau} y_2(\tau)d\tau. \end{aligned} \tag{3-151}$$

In the following we will determine $p_1(0)$ and $p_2(0)$. By inserting (3-149), (3-150) and (3-151) into (3-144) and using the Fubini theorem (see Theorem 1.49) it gives (assume $\Re z + \mu > 0$)

$$\begin{aligned} p_1(0) = {} & \lambda\eta_1\left\{\frac{y_0}{z+\lambda\eta_1} + \frac{\mu\eta_2\left[p_1(0) + \int_0^\infty y_1(x)dx\right]}{(z+\mu+\lambda\eta_1)(z+\lambda\eta_1)}\right\} \\ & + \mu q_2\left\{\frac{p_1(0)}{z+\mu+\lambda\eta_1} + \frac{1}{z+\mu+\lambda\eta_1}\int_0^\infty y_1(x)dx\right\} \\ & + \mu\eta_2\left\{\frac{p_2(0)}{z+\mu} + p_1(0)\left[\frac{1}{z+\mu} - \frac{1}{z+\mu+\lambda\eta_1}\right]\right. \\ & + \frac{1}{z+\mu}\int_0^\infty y_1(\xi)d\xi - \frac{1}{z+\mu+\lambda\eta_1}\int_0^\infty y_1(\xi)d\xi \end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{z+\mu}\int_0^\infty y_2(x)dx\Bigg\} \\
&=\frac{\lambda\eta_1}{z+\lambda\eta_1}y_0+\frac{\mu\eta_2}{z+\mu}p_2(0)+\frac{\mu\eta_2}{z+\mu}\int_0^\infty y_2(x)dx \\
&\quad+\left[\frac{\lambda\eta_1\mu\eta_2}{(z+\lambda\eta_1)(z+\mu+\lambda\eta_1)}+\frac{\mu q_2}{z+\mu+\lambda\eta_1}\right. \\
&\quad\left.+\frac{\lambda\eta_1\mu\eta_2}{(z+\mu+\lambda\eta_1)(z+\mu)}\right]p_1(0) \\
&\quad+\left[\frac{\lambda\eta_1\mu\eta_2}{(z+\lambda\eta_1)(z+\mu+\lambda\eta_1)}+\frac{\mu q_2}{z+\mu+\lambda\eta_1}\right. \\
&\quad\left.+\frac{\lambda\eta_1\mu\eta_2}{(z+\mu+\lambda\eta_1)(z+\mu)}\right]\int_0^\infty y_1(x)dx \\
&\xRightarrow{(**)} \\
&\frac{z(z+\mu)(z+\lambda\eta_1+\mu\eta_2)+\lambda\eta_1(z+\lambda\eta_1)(z+\mu q_2)}{(z+\mu+\lambda\eta_1)(z+\mu)(z+\lambda\eta_1)}p_1(0) \\
&\quad-\frac{\mu\eta_2}{z+\mu}p_2(0) \\
&=\frac{\lambda\eta_1}{z+\lambda\eta_1}y_0+\frac{\mu\eta_2}{z+\mu}\int_0^\infty y_2(x)dx \\
&\quad+\left\{\frac{\lambda\eta_1\mu\eta_2(z+\mu)+\mu q_2(z+\mu)(z+\lambda\eta_1)}{(z+\lambda\eta_1)(z+\mu)(z+\mu+\lambda\eta_1)}\right. \\
&\quad\left.+\frac{\lambda\eta_1\mu\eta_2(z+\lambda\eta_1)}{(z+\lambda\eta_1)(z+\mu)(z+\mu+\lambda\eta_1)}\right\}\int_0^\infty y_1(x)dx \\
&\xRightarrow{\times(z+\mu+\lambda\eta_1)(z+\mu)(z+\lambda\eta_1)} \\
&[z(z+\mu)(z+\lambda\eta_1+\mu\eta_2)+\lambda\eta_1(z+\lambda\eta_1)(z+\mu q_2)]p_1(0) \\
&\quad-\mu\eta_2(z+\lambda\eta_1)(z+\mu+\lambda\eta_1)p_2(0) \\
&=\lambda\eta_1(z+\mu)(z+\mu+\lambda\eta_1)y_0 \\
&\quad+\mu\eta_2(z+\lambda\eta_1)(z+\mu+\lambda\eta_1)\int_0^\infty y_2(x)dx \\
&\quad+[\lambda\eta_1\mu\eta_2(z+\lambda\eta_1) \\
&\qquad+(z\mu q_2+\lambda\eta_1\mu)(z+\mu)]\int_0^\infty y_1(x)dx. \qquad (3\text{-}152)
\end{aligned}
$$

During calculation of the coefficient of $p_1(0)$ in (3-152) we have applied the following equality, noting $q_2+\eta_2=1$:

$$
\begin{aligned}
&(z+\lambda\eta_1)(z+\mu)(z+\mu+\lambda\eta_1)-[\lambda\eta_1\mu\eta_2(z+\mu) \\
&\quad+\mu q_2(z+\lambda\eta_1)(z+\mu)+\lambda\eta_1\mu\eta_2(z+\lambda\eta_1)]
\end{aligned}
$$

$$\begin{aligned}
&= (z+\lambda\eta_1)(z+\mu)(z+\mu+\lambda\eta_1-\mu q_2) \\
&\quad - \lambda\eta_1\mu\eta_2(z+\mu) - \lambda\eta_1\mu\eta_2(z+\lambda\eta_1) \\
&= (z+\lambda\eta_1)(z+\mu)(z+\lambda\eta_1+\mu\eta_2) \\
&\quad - \lambda\eta_1\mu\eta_2(z+\mu) - \lambda\eta_1\mu\eta_2(z+\lambda\eta_1) \\
&= z(z+\mu)(z+\lambda\eta_1+\mu\eta_2) + \lambda\eta_1(z+\mu)(z+\lambda\eta_1+\mu\eta_2) \\
&\quad - \lambda\eta_1\mu\eta_2(z+\mu) - \lambda\eta_1\mu\eta_2(z+\lambda\eta_1) \\
&= z(z+\mu)(z+\lambda\eta_1+\mu\eta_2) \\
&\quad + \lambda\eta_1(z+\mu)[z+\lambda\eta_1+\mu\eta_2-\mu\eta_2] - \lambda\eta_1\mu\eta_2(z+\lambda\eta_1) \\
&= z(z+\mu)(z+\lambda\eta_1+\mu\eta_2) \\
&\quad + \lambda\eta_1(z+\mu)(z+\lambda\eta_1) - \lambda\eta_1\mu\eta_2(z+\lambda\eta_1) \\
&= z(z+\mu)(z+\lambda\eta_1+\mu\eta_2) + \lambda\eta_1(z+\lambda\eta_1)(z+\mu-\mu\eta_2) \\
&= z(z+\mu)(z+\lambda\eta_1+\mu\eta_2) + \lambda\eta_1(z+\lambda\eta_1)(z+\mu q_2). \qquad (**)
\end{aligned}$$

By inserting (3-151) into (3-145) and using the Fubini theorem (see Theorem 1.49) and $q_2+\eta_2=1$ we have

$$\begin{aligned}
p_2(0) &= \frac{\mu q_2}{z+\mu}p_2(0) + \frac{\lambda\eta_1\mu q_2}{(z+\mu+\lambda\eta_1)(z+\mu)}p_1(0) \\
&\quad + \frac{\lambda\eta_1\mu q_2}{(z+\mu+\lambda\eta_1)(z+\mu)}\int_0^\infty y_1(x)dx \\
&\quad + \frac{\mu q_2}{z+\mu}\int_0^\infty y_2(x)dx \\
&\Longrightarrow \\
&\quad \frac{\lambda\eta_1\mu q_2}{(z+\mu+\lambda\eta_1)(z+\mu)}p_1(0) - \frac{(z+\mu\eta_2)}{z+\mu}p_2(0) \\
&= -\frac{\lambda\eta_1\mu q_2}{(z+\mu+\lambda\eta_1)(z+\mu)}\int_0^\infty y_1(x)dx \\
&\quad - \frac{\mu q_2}{z+\mu}\int_0^\infty y_2(x)dx \\
&\xRightarrow{\times(z+\mu+\lambda\eta_1)(z+\mu)} \\
&\lambda\eta_1\mu q_2 p_1(0) - (z+\mu\eta_2)(z+\mu+\lambda\eta_1)p_2(0) \\
&= -\lambda\eta_1\mu q_2\int_0^\infty y_1(x)dx \\
&\quad - \mu q_2(z+\mu+\lambda\eta_1)\int_0^\infty y_2(x)dx. \qquad (3\text{-}153)
\end{aligned}$$

From (3-152), (3-153), $q_2+\eta_2=1$ and the Cramer rule we calculate

$$\begin{pmatrix} p_1(0) \\ p_2(0) \end{pmatrix} = \begin{pmatrix} \frac{|\mathcal{D}|}{|C|} \\ \frac{|\mathcal{F}|}{|C|} \end{pmatrix},$$

where

$$C = \begin{pmatrix} z(z+\mu)(z+\lambda\eta_1+\mu\eta_2)+\lambda\eta_1(z+\lambda\eta_1)(z+\mu q_2) \\ \lambda\eta_1\mu q_2 \end{pmatrix.$$
$$\begin{matrix} -\mu\eta_2(z+\lambda\eta_1)(z+\mu+\lambda\eta_1) \\ -(z+\mu\eta_2)(z+\mu+\lambda\eta_1) \end{matrix}\Bigg)$$

$$\begin{aligned}
&\Longrightarrow \\
|C| &= -(z+\mu\eta_2)(z+\mu+\lambda\eta_1) \\
&\quad \times [z(z+\mu)(z+\lambda\eta_1+\mu\eta_2)+\lambda\eta_1(z+\lambda\eta_1)(z+\mu q_2)] \\
&\quad + \lambda\eta_1\mu q_2\mu\eta_2(z+\lambda\eta_1)(z+\mu+\lambda\eta_1) \\
&= -z(z+\mu\eta_2)(z+\mu+\lambda\eta_1)(z+\mu)(z+\lambda\eta_1+\mu\eta_2) \\
&\quad - \lambda\eta_1(z+\mu\eta_2)(z+\mu+\lambda\eta_1)(z+\lambda\eta_1)(z+\mu q_2) \\
&\quad + \lambda\eta_1\mu q_2\mu\eta_2(z+\lambda\eta_1)(z+\mu+\lambda\eta_1) \\
&= -z(z+\mu\eta_2)(z+\mu+\lambda\eta_1)(z+\mu)(z+\lambda\eta_1+\mu\eta_2) \\
&\quad - z\lambda\eta_1(z+\lambda\eta_1)(z+\mu+\lambda\eta_1)(z+\mu) \\
&= -z(z+\mu)(z+\mu+\lambda\eta_1) \\
&\quad \times \left[(z+\lambda\eta_1+\mu\eta_2)^2-\lambda\eta_1\mu\eta_2\right],
\end{aligned} \tag{3-154}$$

$$\begin{aligned}
\mathcal{D} &= \begin{pmatrix} \lambda\eta_1(z+\mu+\lambda\eta_1)(z+\mu)y_0 & 0 \\ -\lambda\eta_1\mu q_2\int_0^\infty y_1(x)dx & 0 \end{pmatrix} \\
&\quad + \begin{pmatrix} \mu\eta_2(z+\lambda\eta_1)(z+\mu+\lambda\eta_1)\int_0^\infty y_2(x)dx & 0 \\ -\mu q_2(z+\mu+\lambda\eta_1)\int_0^\infty y_2(x)dx & 0 \end{pmatrix} \\
&\quad + \begin{pmatrix} \lambda\eta_1\mu\eta_2(z+\lambda\eta_1)\int_0^\infty y_1(x)dx & 0 \\ 0 & 0 \end{pmatrix} \\
&\quad + \begin{pmatrix} (z+\mu)(z\mu q_2+\lambda\eta_1\mu)\int_0^\infty y_1(x)dx & 0 \\ 0 & 0 \end{pmatrix} \\
&\quad + \begin{pmatrix} 0 & -\mu\eta_2(z+\lambda\eta_1)(z+\mu+\lambda\eta_1) \\ 0 & -(z+\mu\eta_2)(z+\mu+\lambda\eta_1) \end{pmatrix}
\end{aligned}$$

$$\begin{aligned}
&\Longrightarrow \\
|\mathcal{D}| &= -(z+\mu\eta_2)(z+\mu+\lambda\eta_1)\Bigg\{\lambda\eta_1(z+\mu+\lambda\eta_1)(z+\mu)y_0 \\
&\quad + \mu\eta_2(z+\lambda\eta_1)(z+\mu+\lambda\eta_1)\int_0^\infty y_2(x)dx \\
&\quad + [\lambda\eta_1\mu\eta_2(z+\lambda\eta_1)+(z+\mu)(z\mu q_2+\lambda\eta_1\mu)]\int_0^\infty y_1(x)dx\Bigg\} \\
&\quad + \mu\eta_2(z+\lambda\eta_1)(z+\mu+\lambda\eta_1)\Bigg\{-\lambda\eta_1\mu q_2\int_0^\infty y_1(x)dx
\end{aligned}$$

$$
\begin{aligned}
&- \mu q_2(z+\mu+\lambda\eta_1)\int_0^\infty y_2(x)dx\bigg\} \\
&= -\lambda\eta_1(z+\mu\eta_2)(z+\mu)(z+\mu+\lambda\eta_1)^2 y_0 \\
&\quad - \mu\eta_2(z+\mu\eta_2)(z+\lambda\eta_1)(z+\mu+\lambda\eta_1)^2\int_0^\infty y_2(x)dx \\
&\quad - (z+\mu\eta_2)(z+\mu+\lambda\eta_1)[\lambda\eta_1\mu\eta_2(z+\lambda\eta_1) \\
&\qquad + (z+\mu)(z\mu q_2+\lambda\eta_1\mu)]\int_0^\infty y_1(x)dx \\
&\quad - \mu\eta_2(z+\lambda\eta_1)(z+\mu+\lambda\eta_1)\left[\lambda\eta_1\mu q_2\int_0^\infty y_1(x)dx\right. \\
&\quad \left. +\mu q_2(z+\mu+\lambda\eta_1)\int_0^\infty y_2(x)dx\right] \\
&= -\lambda\eta_1(z+\mu\eta_2)(z+\mu)(z+\mu+\lambda\eta_1)^2 y_0 \\
&\quad - \mu\eta_2(z+\mu\eta_2)(z+\lambda\eta_1)(z+\mu+\lambda\eta_1)^2\int_0^\infty y_2(x)dx \\
&\quad - \mu\eta_2\mu q_2(z+\lambda\eta_1)(z+\mu+\lambda\eta_1)^2\int_0^\infty y_2(x)dx \\
&\quad - (z+\mu\eta_2)(z+\mu+\lambda\eta_1)[\lambda\eta_1\mu\eta_2(z+\lambda\eta_1) \\
&\quad + (z+\mu)(z\mu q_2+\lambda\eta_1\mu)]\int_0^\infty y_1(x)dx \\
&\quad - \lambda\eta_1\mu q_2\mu\eta_2(z+\lambda\eta_1)(z+\mu+\lambda\eta_1)\int_0^\infty y_1(x)dx \\
&= -\lambda\eta_1(z+\mu\eta_2)(z+\mu)(z+\mu+\lambda\eta_1)^2 y_0 \\
&\quad - \mu\eta_2(z+\lambda\eta_1)(z+\mu+\lambda\eta_1)^2(z+\mu\eta_2+\mu q_2)\int_0^\infty y_2(x)dx \\
&\quad - \lambda\eta_1\mu\eta_2(z+\lambda\eta_1)(z+\mu+\lambda\eta_1)(z+\mu\eta_2+\mu q_2)\int_0^\infty y_1(x)dx \\
&\quad - (z+\mu\eta_2)(z+\mu+\lambda\eta_1)(z+\mu)(z\mu q_2+\lambda\eta_1\mu)\int_0^\infty y_1(x)dx \\
&= -\lambda\eta_1(z+\mu)(z+\mu\eta_2)(z+\mu+\lambda\eta_1)^2 y_0 \\
&\quad - \mu\eta_2(z+\mu)(z+\lambda\eta_1)(z+\mu+\lambda\eta_1)^2\int_0^\infty y_2(x)dx \\
&\quad - (z+\mu)(z+\mu+\lambda\eta_1)[\lambda\eta_1\mu\eta_2(z+\lambda\eta_1) \\
&\quad + (z+\mu\eta_2)(z\mu q_2+\lambda\eta_1\mu)]\int_0^\infty y_1(x)dx,
\end{aligned} \tag{3-155}
$$

$$
\mathcal{F} = \begin{pmatrix} z(z+\mu)(z+\lambda\eta_1+\mu\eta_2)+\lambda\eta_1(z+\lambda\eta_1)(z+\mu q_2) & 0 \\ \lambda\eta_1\mu q_2 & 0 \end{pmatrix}
$$

$$
\begin{aligned}
&+ \begin{pmatrix} 0 & \lambda\eta_1(z+\mu+\lambda\eta_1)(z+\mu)y_0 \\ 0 & -\lambda\eta_1\mu q_2 \int_0^\infty y_1(x)dx \end{pmatrix} \\
&+ \begin{pmatrix} 0 & \mu\eta_2(z+\lambda\eta_1)(z+\mu+\lambda\eta_1)\int_0^\infty y_2(x)dx \\ 0 & -\mu q_2(z+\mu+\lambda\eta_1)\int_0^\infty y_2(x)dx \end{pmatrix} \\
&+ \begin{pmatrix} 0 & \lambda\eta_1\mu\eta_2(z+\lambda\eta_1)\int_0^\infty y_1(x)dx \\ 0 & 0 \end{pmatrix} \\
&+ \begin{pmatrix} 0 & (z+\mu)(z\mu q_2+\lambda\eta_1\mu)\int_0^\infty y_1(x)dx \\ 0 & 0 \end{pmatrix}
\end{aligned}
$$

$$\Longrightarrow$$

$$
\begin{aligned}
|\mathcal{F}| = & -\lambda\eta_1\mu q_2\big[z(z+\mu)(z+\lambda\eta_1+\mu\eta_2) \\
& + \lambda\eta_1(z+\lambda\eta_1)(z+\mu q_2)\big]\int_0^\infty y_1(x)dx \\
& - \mu q_2(z+\mu+\lambda\eta_1)\left[z(z+\mu)(z+\lambda\eta_1+\mu\eta_2)\right. \\
& \left.+\lambda\eta_1(z+\lambda\eta_1)(z+\mu q_2)\right]\int_0^\infty y_2(x)dx \\
& - (\lambda\eta_1)^2\mu q_2(z+\mu+\lambda\eta_1)(z+\mu)y_0 \\
& - \lambda\eta_1\mu q_2\mu\eta_2(z+\lambda\eta_1)(z+\mu+\lambda\eta_1)\int_0^\infty y_2(x)dx \\
& - \lambda\eta_1\mu q_2\big[\lambda\eta_1\mu\eta_2(z+\lambda\eta_1) \\
& + (z+\mu)(z\mu q_2+\lambda\eta_1\mu)\big]\int_0^\infty y_1(x)dx \\
= & -(\lambda\eta_1)^2\mu q_2(z+\mu)(z+\mu+\lambda\eta_1)y_0 \\
& - \mu q_2(z+\mu+\lambda\eta_1)\big[z(z+\mu)(z+\lambda\eta_1+\mu\eta_2) \\
& + \lambda\eta_1(z+\lambda\eta_1)(z+\mu q_2)+\lambda\eta_1\mu\eta_2(z+\lambda\eta_1)\big]\int_0^\infty y_2(x)dx \\
& - \lambda\eta_1\mu q_2\big[z(z+\mu)(z+\lambda\eta_1+\mu\eta_2)+\lambda\eta_1(z+\lambda\eta_1)(z+\mu q_2) \\
& + \lambda\eta_1\mu\eta_2(z+\lambda\eta_1)+(z+\mu)(z\mu q_2+\lambda\eta_1\mu)\big]\int_0^\infty y_1(x)dx \\
= & -(\lambda\eta_1)^2\mu q_2(z+\mu)(z+\mu+\lambda\eta_1)y_0 \\
& - \mu q_2(z+\mu+\lambda\eta_1)\big[z(z+\mu)(z+\lambda\eta_1+\mu\eta_2) \\
& + \lambda\eta_1(z+\lambda\eta_1)(z+\mu)\big]\int_0^\infty y_2(x)dx \\
& - \lambda\eta_1\mu q_2(z+\mu)\big[z(z+\lambda\eta_1+\mu\eta_2) \\
& + \lambda\eta_1(z+\lambda\eta_1)+(z\mu q_2+\lambda\eta_1\mu)\big]\int_0^\infty y_1(x)dx \\
= & -(\lambda\eta_1)^2\mu q_2(z+\mu)(z+\mu+\lambda\eta_1)y_0
\end{aligned}
$$

$$- \mu q_2(z+\mu)(z+\mu+\lambda\eta_1)[(z+\lambda\eta_1)^2 + z\mu\eta_2] \int_0^\infty y_2(x)dx$$
$$- \lambda\eta_1\mu q_2(z+\mu+\lambda\eta_1)(z+\mu)(z+\lambda\eta_1) \int_0^\infty y_1(x)dx. \tag{3-156}$$

(3-154)–(3-157) give

$$\begin{aligned}
p_1(0) &= \frac{|\mathcal{D}|}{|C|} \\
&= \frac{\lambda\eta_1(z+\mu\eta_2)(z+\mu+\lambda\eta_1)}{z[(z+\lambda\eta_1+\mu\eta_2)^2 - \lambda\eta_1\mu\eta_2]} y_0 \\
&\quad + \left\{ \frac{\lambda\eta_1\mu\eta_2(z+\lambda\eta_1)}{z[(z+\lambda\eta_1+\mu\eta_2)^2 - \lambda\eta_1\mu\eta_2]} \right. \\
&\quad \left. + \frac{(z+\mu\eta_2)(z\mu q_2 + \lambda\eta_1\mu)}{z[(z+\lambda\eta_1+\mu\eta_2)^2 - \lambda\eta_1\mu\eta_2]} \right\} \int_0^\infty y_1(x)dx \\
&\quad + \frac{\mu\eta_2(z+\lambda\eta_1)(z+\mu+\lambda\eta_1)}{z[(z+\lambda\eta_1+\mu\eta_2)^2 - \lambda\eta_1\mu\eta_2]} \int_0^\infty y_2(x)dx.
\end{aligned} \tag{3-157}$$

$$\begin{aligned}
p_2(0) &= \frac{|\mathcal{F}|}{|C|} = \frac{\mu q_2(\lambda\eta_1)^2}{z[(z+\lambda\eta_1+\mu\eta_2)^2 - \lambda\eta_1\mu\eta_2]} y_0 \\
&\quad + \frac{\lambda\eta_1\mu q_2(z+\lambda\eta_1)}{z[(z+\lambda\eta_1+\mu\eta_2)^2 - \lambda\eta_1\mu\eta_2]} \int_0^\infty y_1(x)dx \\
&\quad + \frac{\mu q_2[(z+\lambda\eta_1)^2 + z\mu\eta_2]}{z[(z+\lambda\eta_1+\mu\eta_2)^2 - \lambda\eta_1\mu\eta_2]} \int_0^\infty y_2(x)dx.
\end{aligned} \tag{3-158}$$

By inserting (3-157) and (3-158) into (3-149), (3-150) and (3-151) we obtain the result of this lemma. □

Lemma 3.18. *If $\lambda\eta_1 < \mu\eta_2$, then $\mathbb{P}$, the projection operator in Theorem 3.13, can be expressed as*

$$\mathbb{P}y = \begin{pmatrix} \chi_1 \\ \chi_2 \\ \chi_3 \end{pmatrix}, \quad \forall y \in X,$$

where

$$\begin{aligned}
\chi_1 &= \frac{(\mu\eta_2)^2}{(\lambda\eta_1+\mu\eta_2)^2 - \lambda\eta_1\mu\eta_2} \\
&\quad \times \left[y_0 + \int_0^\infty y_1(x)dx + \int_0^\infty y_2(x)dx \right], \\
\chi_2 &= \frac{\lambda\eta_1\mu\eta_2(\mu+\lambda\eta_1)}{(\lambda\eta_1+\mu\eta_2)^2 - \lambda\eta_1\mu\eta_2} \\
&\quad \times e^{-(\mu+\lambda\eta_1)x} \left[y_0 + \int_0^\infty y_1(x)dx + \int_0^\infty y_2(x)dx \right],
\end{aligned}$$

$$\chi_3 = \left[\frac{\mu\lambda\eta_1(\lambda\eta_1+\mu\eta_2)}{(\lambda\eta_1+\mu\eta_2)^2-\lambda\eta_1\mu\eta_2}e^{-\mu x}\right.$$
$$\left. -\frac{\lambda\eta_1\mu\eta_2(\mu+\lambda\eta_1)}{(\lambda\eta_1+\mu\eta_2)^2-\lambda\eta_1\mu\eta_2}e^{-(\mu+\lambda\eta_1)x}\right]$$
$$\times\left[y_0+\int_0^\infty y_1(x)dx+\int_0^\infty y_2(x)dx\right].$$

Especially, for $p(0) = (1,0,0)$*, the initial value of the system* (3-15)*, we have*

$$\mathbb{P}p(0) = \begin{pmatrix} \frac{(\mu\eta_2)^2}{(\lambda\eta_1+\mu\eta_2)^2-\lambda\eta_1\mu\eta_2} \\ \frac{\lambda\eta_1\mu\eta_2(\mu+\lambda\eta_1)}{(\lambda\eta_1+\mu\eta_2)^2-\lambda\eta_1\mu\eta_2}e^{-(\mu+\lambda\eta_1)x} \\ \frac{\mu\lambda\eta_1(\lambda\eta_1+\mu\eta_2)}{(\lambda\eta_1+\mu\eta_2)^2-\lambda\eta_1\mu\eta_2}e^{-\mu x} - \frac{\lambda\eta_1\mu\eta_2(\mu+\lambda\eta_1)}{(\lambda\eta_1+\mu\eta_2)^2-\lambda\eta_1\mu\eta_2}e^{-(\mu+\lambda\eta_1)x} \end{pmatrix} = p(x).$$

Here $p(x)$ *is the steady-state solution of the system* (3-15)*, that is, the eigenvector in Lemma* 3.10*.*

Proof. Since Lemma 3.10, Lemma 3.12, Lemma 3.15 and Lemma 3.16 show that $z = 0$ is a pole of $(zI - A - U - E)^{-1}$ of order 1, from the residue theorem and Lemma 3.17 it follows that, $\forall y \in X$,

$$\begin{aligned}\mathbb{P}y &= \frac{1}{2\pi i}\int_\Gamma (zI-A-U-E)^{-1}ydz \\ &= \lim_{z\to 0} z(zI-A-U-E)^{-1}y \\ &= \lim_{z\to 0} z\begin{pmatrix}\psi_1\\ \psi_2\\ \psi_3\end{pmatrix} = \begin{pmatrix}\lim_{z\to 0} z\psi_1\\ \lim_{z\to 0} z\psi_2\\ \lim_{z\to 0} z\psi_3\end{pmatrix}.\end{aligned} \tag{3-159}$$

From the definition of ψ_i $(i = 1, 2, 3)$ (see Lemma 3.17) we calculate

$$\begin{aligned}\lim_{z\to 0} z\psi_1 &= \lim_{z\to 0}\frac{z}{(z+\lambda\eta_1)}y_0 \\ &\quad + \lim_{z\to 0}\frac{\mu\eta_2 z}{(z+\lambda\eta_1)(z+\mu+\lambda\eta_1)}\int_0^\infty y_1(x)dx \\ &\quad + \lim_{z\to 0}\frac{\lambda\eta_1\mu\eta_2(z+\mu\eta_2)}{(z+\lambda\eta_1)[(z+\lambda\eta_1+\mu\eta_2)^2-\lambda\eta_1\mu\eta_2]}y_0 \\ &\quad + \lim_{z\to 0}\left\{\frac{\mu\eta_2\lambda\eta_1\mu\eta_2}{(z+\mu+\lambda\eta_1)[(z+\lambda\eta_1+\mu\eta_2)^2-\lambda\eta_1\mu\eta_2]}\right. \\ &\quad \left. + \frac{\mu\eta_2(z\mu q_2+\lambda\eta_1\mu)(z+\mu\eta_2)}{(z+\lambda\eta_1)(z+\mu+\lambda\eta_1)[(z+\lambda\eta_1+\mu\eta_2)^2-\lambda\eta_1\mu\eta_2]}\right\} \\ &\quad \times\int_0^\infty y_1(x)dx\end{aligned}$$

$$
\begin{aligned}
&+ \lim_{z\to 0} \frac{(\mu\eta_2)^2}{[(z+\lambda\eta_1+\mu\eta_2)^2-\lambda\eta_1\mu\eta_2]} \int_0^\infty y_2(x)dx \\
&= \frac{(\mu\eta_2)^2}{(\lambda\eta_1+\mu\eta_2)^2-\lambda\eta_1\mu\eta_2} \\
&\quad \times \left[y_0 + \int_0^\infty y_1(x)dx + \int_0^\infty y_2(x)dx \right] \\
&:= \chi_1.
\end{aligned} \tag{3-160}
$$

$$
\begin{aligned}
\lim_{z\to 0} z\psi_2 &= \lim_{z\to 0} z e^{-(z+\mu+\lambda\eta_1)x} \int_0^x e^{(z+\mu+\lambda\eta_1)\tau} y_1(\tau)d\tau \\
&+ \lim_{z\to 0} \frac{\lambda\eta_1(z+\mu\eta_2)(z+\mu+\lambda\eta_1)}{[(z+\lambda\eta_1+\mu\eta_2)^2-\lambda\eta_1\mu\eta_2]} e^{-(z+\mu+\lambda\eta_1)x} y_0 \\
&+ \lim_{z\to 0} \frac{(\lambda\eta_1)^2\mu\eta_2+(z+\mu\eta_2)(z\mu q_2+\lambda\eta_1\mu)}{[(z+\lambda\eta_1+\mu\eta_2)^2-\lambda\eta_1\mu\eta_2]} \\
&\quad \times e^{-(z+\mu+\lambda\eta_1)x} \int_0^\infty y_1(x)dx \\
&+ \lim_{z\to 0} \frac{\mu\eta_2(z+\lambda\eta_1)(z+\mu+\lambda\eta_1)}{[(z+\lambda\eta_1+\mu\eta_2)^2-\lambda\eta_1\mu\eta_2]} e^{-(z+\mu+\lambda\eta_1)x} \int_0^\infty y_2(x)dx \\
&= \frac{\lambda\eta_1\mu\eta_2(\mu+\lambda\eta_1)}{(\lambda\eta_1+\mu\eta_2)^2-\lambda\eta_1\mu\eta_2} \\
&\quad \times e^{-(\mu+\lambda\eta_1)x} \left[y_0 + \int_0^\infty y_1(x)dx + \int_0^\infty y_2(x)dx \right] \\
&:= \chi_2.
\end{aligned} \tag{3-161}
$$

$$
\begin{aligned}
\lim_{z\to 0} z\psi_3 &= \lim_{z\to 0} z e^{-(z+\mu)x} \int_0^x e^{(z+\mu)\xi} y_1(\xi)d\xi \\
&- \lim_{z\to 0} z e^{-(z+\mu+\lambda\eta_1)x} \int_0^x e^{(z+\mu+\lambda\eta_1)\xi} y_1(\xi)d\xi \\
&+ \lim_{z\to 0} z e^{-(z+\mu)x} \int_0^x e^{(z+\mu)\tau} y_2(\tau)d\tau \\
&+ \lim_{z\to 0} \frac{(\lambda\eta_1)^2\mu q_2+\lambda\eta_1(z+\mu\eta_2)(z+\mu+\lambda\eta_1)}{[(z+\lambda\eta_1+\mu\eta_2)^2-\lambda\eta_1\mu\eta_2]} e^{-(z+\mu)x} y_0 \\
&- \lim_{z\to 0} \frac{\lambda\eta_1(z+\mu\eta_2)(z+\mu+\lambda\eta_1)}{[(z+\lambda\eta_1+\mu\eta_2)^2-\lambda\eta_1\mu\eta_2]} e^{-(z+\mu+\lambda\eta_1)x} y_0 \\
&+ \lim_{z\to 0} \left\{ \frac{\lambda\eta_1\mu q_2(z+\lambda\eta_1)+\lambda\eta_1\mu\eta_2(z+\lambda\eta_1)}{[(z+\lambda\eta_1+\mu\eta_2)^2-\lambda\eta_1\mu\eta_2]} \right. \\
&+ \left. \frac{(z\mu q_2+\lambda\eta_1\mu)(z+\mu\eta_2)}{[(z+\lambda\eta_1+\mu\eta_2)^2-\lambda\eta_1\mu\eta_2]} \right\} e^{-(z+\mu)x} \int_0^\infty y_1(x)dx
\end{aligned}
$$

$$
\begin{aligned}
& - \lim_{z\to 0} \frac{[\lambda\eta_1\mu\eta_2(z+\lambda\eta_1)+(z\mu q_2+\lambda\eta_1\mu)(z+\mu\eta_2)]}{[(z+\lambda\eta_1+\mu\eta_2)^2-\lambda\eta_1\mu\eta_2]} \\
& \times e^{-(z+\mu+\lambda\eta_1)x} \int_0^\infty y_1(x)dx \\
& + \lim_{z\to 0} \left\{ \frac{\mu q_2[(z+\lambda\eta_1)^2+z\mu\eta_2]}{(z+\mu\eta_2)[(z+\lambda\eta_1+\mu\eta_2)^2-\lambda\eta_1\mu\eta_2]} \right. \\
& + \left. \frac{\mu\eta_2(z+\lambda\eta_1)(z+\mu+\lambda\eta_1)}{[(z+\lambda\eta_1+\mu\eta_2)^2-\lambda\eta_1\mu\eta_2]} \right\} \\
& \quad \times e^{-(z+\mu)x} \int_0^\infty y_2(x)dx \\
& - \lim_{z\to 0} \frac{\mu\eta_2(z+\lambda\eta_1)(z+\mu+\lambda\eta_1)}{[(z+\lambda\eta_1+\mu\eta_2)^2-\lambda\eta_1\mu\eta_2]} \\
& \quad \times e^{-(z+\mu+\lambda\eta_1)x} \int_0^\infty y_2(x)dx \\
= & \left[\frac{\mu\lambda\eta_1(\lambda\eta_1+\mu\eta_2)}{(\lambda\eta_1+\mu\eta_2)^2-\lambda\eta_1\mu\eta_2} e^{-\mu x} \right. \\
& \left. - \frac{\lambda\eta_1\mu\eta_2(\mu+\lambda\eta_1)}{(\lambda\eta_1+\mu\eta_2)^2-\lambda\eta_1\mu\eta_2} e^{-(\mu+\lambda\eta_1)x} \right] \\
& \quad \times \left[y_0 + \int_0^\infty y_1(x)dx + \int_0^\infty y_2(x)dx \right] \\
:= & \chi_3. \qquad (3\text{-}162)
\end{aligned}
$$

Particularly, for $p(0) = (1, 0, 0)$, the initial value of the system (3-15), (3-160)–(3-162) become

$$
\chi_1\big|_{p(0)} = \frac{(\mu\eta_2)^2}{(\lambda\eta_1+\mu\eta_2)^2-\lambda\eta_1\mu\eta_2}, \qquad (3\text{-}163)
$$

$$
\chi_2\big|_{p(0)} = \frac{\lambda\eta_1\mu\eta_2(\mu+\lambda\eta_1)}{(\lambda\eta_1+\mu\eta_2)^2-\lambda\eta_1\mu\eta_2} e^{-(\mu+\lambda\eta_1)x}, \qquad (3\text{-}164)
$$

$$
\begin{aligned}
\chi_3\big|_{p(0)} = & \frac{\mu\lambda\eta_1(\lambda\eta_1+\mu\eta_2)}{(\lambda\eta_1+\mu\eta_2)^2-\lambda\eta_1\mu\eta_2} e^{-\mu x} \\
& - \frac{\lambda\eta_1\mu\eta_2(\mu+\lambda\eta_1)}{(\lambda\eta_1+\mu\eta_2)^2-\lambda\eta_1\mu\eta_2} e^{-(\mu+\lambda\eta_1)x}. \qquad (3\text{-}165)
\end{aligned}
$$

By inserting (3-160)–(3-162) into (3-159) and (3-163)–(3-165) into (3-159) respectively we obtain the results of this lemma. □

Through combining Theorem 3.13 with Lemma 3.18 we derive

Theorem 3.19. *If* $\lambda\eta_1 < \mu\eta_2$, *then the time-dependent solution* $p(x,t)$ *of the system* (3-15) *converges exponentially to its steady-state solution* $p(x)$, *that is, there exist* *positive numbers* $\delta > 0$ *and* $\mathcal{M} \geq 1$ *such that*

$$\|p(\cdot,t) - p(\cdot)\| \leq \mathcal{M}e^{-\delta t}, \quad \forall t \in [0,\infty).$$

Proof. Since $p(0) = (1,0,0) \in D(A)$, Theorem 3.3 implies that the system (3-15) has a unique positive time-dependent solution $p(x,t)$ which can be expressed as (see (3-39))

$$p(x,t) = T(t)p(0) = T(t)(1,0,0), \quad \forall t \in [0,\infty). \tag{3-166}$$

By Lemma 3.18

$$p(x) = \mathbb{P}p(0). \tag{3-167}$$

By combining Theorem 3.13 with (3-166) and (3-167) we have

$$\begin{aligned}\|p(\cdot,t) - p(\cdot)\| &= \|T(t)p(0) - \mathbb{P}p(0)\| \leqslant \|T(t) - \mathbb{P}\|\|p(0)\| \\ &\leq \mathcal{M}e^{-\delta t}\|p(0)\| = \mathcal{M}e^{-\delta t}, \quad \forall\, t \in [0,\infty).\end{aligned}$$

□

Remark 3.20. All the above results in this chapter hold for the system consisting of a reliable machine, an unreliable machine and a storage buffer with finite capacity, the detail discussion one can find in Gupur et al. [63], Haji et al. [68] and Miao et al. [87].

3.4 Asymptotic Behavior of Some Reliability Indices of the System (3-15)

According to Li and Cao [78], the reprocessed rate of machine 1, denoted by $W_{r,1}(t)$, is

$$W_{r,1}(t) = \lambda q_1 p_0(t) + \lambda q_1 \int_0^\infty p_1(x,t)dx. \tag{3-168}$$

$W_{0,1}(t)$, the output rate of machine 1, is denoted by

$$W_{0,1}(t) = \lambda \eta_1 p_0(t) + \lambda \eta_1 \int_0^\infty p_1(x,t)dx. \tag{3-169}$$

The production rate of the system $W(t)$ is defined by

$$W(t) = \eta_1 \int_0^\infty [p_1(x,t) + p_2(x,t)]\mu(x)dx. \tag{3-170}$$

$W_{r,2}(t)$, the reprocessed rate of machine 2, is

$$W_{r,2}(t) = q_2 \int_0^\infty [p_1(x,t) + p_2(x,t)]\mu(x)dx. \tag{3-171}$$

Theorem 3.14 gives

$$\lim_{t\to\infty} p_0(t) = p_0, \quad \lim_{t\to\infty} \int_0^\infty |p_1(x,t) - p_1(x)|dx = 0,$$
$$\lim_{t\to\infty} \int_0^\infty |p_2(x,t) - p_2(x)|dx = 0.$$

Which imply

$$\begin{aligned}
&0 \le \lim_{t\to\infty} \int_0^\infty |\mu(x)p_1(x,t) - \mu(x)p_1(x)|dx \\
&\le \overline{\mu} \lim_{t\to\infty} \int_0^\infty |p_1(x,t) - p_1(x)|dx = 0 \\
&\Longrightarrow \\
&\lim_{t\to\infty} \int_0^\infty |\mu(x)p_1(x,t) - \mu(x)p_1(x)|dx = 0. \qquad (3\text{-}172)
\end{aligned}$$

$$\begin{aligned}
&0 \le \lim_{t\to\infty} \int_0^\infty |\mu(x)p_2(x,t) - \mu(x)p_2(x)|dx \\
&\le \overline{\mu} \lim_{t\to\infty} \int_0^\infty |p_2(x,t) - p_2(x)|dx = 0 \\
&\Longrightarrow \\
&\lim_{t\to\infty} \int_0^\infty |\mu(x)p_2(x,t) - \mu(x)p_2(x)|dx = 0. \qquad (3\text{-}173)
\end{aligned}$$

From (3-172) and (3-173) together with (3-89), (3-90), (3-91) and (3-92) we obtain the asymptotic behavior of the above indices.

Theorem 3.21. *If $\mu(x)$ is Lipschitz continuous and satisfies $0 < \underline{\mu} \le \mu(x) \le \overline{\mu} < \infty$, then*

$$\begin{aligned}
W_{r,1} &:= \lim_{t\to\infty} W_{r,1}(t) \\
&= \lambda q_1 \left\{ 1 + \frac{\lambda\eta_1 \int_0^\infty e^{-\lambda\eta_1 x - \int_0^x \mu(\tau)d\tau}dx}{\eta_2 \int_0^\infty \mu(x)e^{-\lambda\eta_1 x - \int_0^x \mu(\tau)d\tau}dx} \right\} p_0. \\
W_{0,1} &:= \lim_{t\to\infty} W_{0,1}(t) \\
&= \lambda\eta_1 \left\{ 1 + \frac{\lambda\eta_1 \int_0^\infty e^{-\lambda\eta_1 x - \int_0^x \mu(\tau)d\tau}dx}{\eta_2 \int_0^\infty \mu(x)e^{-\lambda\eta_1 x - \int_0^x \mu(\tau)d\tau}dx} \right\} p_0 \\
W &:= \lim_{t\to\infty} W(t) \\
&= \frac{\lambda\eta_1^2}{\eta_2} p_0 + \eta_1 \frac{(\lambda\eta_1)^2 q_2 \int_0^\infty e^{-\lambda\eta_1 x - \int_0^x \mu(\tau)d\tau}dx}{\eta_2^2 \int_0^\infty \mu(x)e^{-\lambda\eta_1 x - \int_0^x \mu(\tau)d\tau}dx} p_0
\end{aligned}$$

$$
\begin{aligned}
&+\eta_1 \frac{\lambda\eta_1\eta_2\left[1-\int_0^\infty \mu(x)e^{-\lambda\eta_1 x-\int_0^x \mu(\tau)d\tau}dx\right]}{\eta_2^2\int_0^\infty \mu(x)e^{-\lambda\eta_1 x-\int_0^x \mu(\tau)d\tau}dx}p_0.\\
W_{r,2} :=& \lim_{t\to\infty} W_{r,2}(t)\\
=& \frac{\lambda\eta_1 q_1}{\eta_2}p_0 + q_1\frac{(\lambda\eta_1)^2 q_2\int_0^\infty e^{-\lambda\eta_1 x-\int_0^x \mu(\tau)d\tau}dx}{\eta_2^2\int_0^\infty \mu(x)e^{-\lambda\eta_1 x-\int_0^x \mu(\tau)d\tau}dx}p_0\\
&+q_1 \frac{\lambda\eta_1\eta_2\left[1-\int_0^\infty \mu(x)e^{-\lambda\eta_1 x-\int_0^x \mu(\tau)d\tau}dx\right]}{\eta_2^2\int_0^\infty \mu(x)e^{-\lambda\eta_1 x-\int_0^x \mu(\tau)d\tau}dx}p_0.
\end{aligned}
$$

Remark 3.22. If we add the normalizing condition

$$p_0+\int_0^\infty p_1(x)dx+\int_0^\infty p_2(x)dx=1,$$

then in the steady-state case we obtain the results in Li and Cao [78]. In other words, our results imply the results in Li and Cao [78].

Notes

By using the idea and methods introduced in this chapter we have obtained the well-posedness and asymptotic behavior of the time-dependent solution of many reliability models, see Gupur [54], Gupur [55], Gupur [56], Gupur [58], Gupur [59], Gupur et al. [61], Gupur et al. [67], Gupur et al. [62], Gupur et al. [64], Ehmet and Gupur [26], Haji and Gupur [68], Miao and Gupur [87], Zhou and Gupur [118], Mao [86], Zhong et al. [117]. In other words, our method and idea are suitable for almost all reliability models which were described by a finite number of partial differential equations with integral boundary conditions.

In 2005, Xu et al. [110] proved that the time-dependent solution of a reliability model strongly converges to its steady-state solution. In 2006, Radl [97] obtained that the time-dependent solution of the system consisting of a reliable machine, an unreliable machine and a storage buffer with finite capacity strongly converges to its steady-state solution (see also Haji and Radl [69]). In 2007, Hu et al. [70] deduced that the time-dependent solution of a reliability model exponentially converges to its steady-state solution.

Chapter 4

Transfer Line Consisting of a Reliable Machine, an Unreliable Machine and a Storage Buffer with Infinite Capacity

In this chapter we will discuss a transfer line consisting of a reliable machine, an unreliable machine and a storage buffer with infinite capacity. This transfer line is useful for a computer integrated manufacturing system (CIMS) and therefore several researchers studied special cases of this system, see Shu [99], Tan [103], Gershwin et al. [37], Yeralan et al. [114] for instance. The most general case was researched by Li and Cao [78]. In 1998, Li and Cao [78] first established a mathematical model of the system by using the supplementary variable technique and studied the steady-state solution and the steady-state reliability indices under the following hypothesis:

$$\lim_{t\to\infty} p(x,t) = p(x),$$

where

$$\begin{aligned} p(x,t) &= (p_0(t), p_1(x,t), p_2(x,t), \dots) - \text{time-dependent solution,} \\ p(x) &= (p_0(t), p_1(x), p_2(x), \dots,) - \text{steady-state solution.} \end{aligned}$$

But, they did not answer the question of whether the above hypothesis holds. Through reading their paper we find that the above hypothesis, in fact, implies the following two hypotheses:

Hypothesis 1. The model has a unique time-dependent solution.

Hypothesis 2. The time-dependent solution converges to its steady-state solution.

In this chapter we will study the above two hypotheses and asymptotic behavior of the reliability indices for the system. Firstly, by using the supplementary variable technique we will establish a mathematical model of the transfer line and convert the model into an abstract Cauchy problem in a Banach space. Next we will consider well-posedness of the problem and prove that the model has a unique

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positive time-dependent solution. Thirdly, when the hazard rate is a constant, by using the idea in Gupur et al. [65] we will obtain the asymptotic behavior of the time-dependent solution of the model and prove that our result is best. Fourthly, by using Greiner's idea we will deduce the asymptotic behavior of the time-dependent solution of the model when the hazard rate is a function and show that our result is the best result. Lastly, we will discuss asymptotic behavior of some reliability indices for the system. One may find the main results in this chapters in Li et al. [78], Gupur et al. [63], Gupur [58], Zhu et al. [119], Wu et al. [107], Gupur et al. [66].

4.1 The Mathematical Model of the Transfer Line Consisting of a Reliable Machine, an Unreliable Machine and a Storage Buffer with Infinite Capacity

An important class of systems is one where objects move sequentially from one station to another, and where they rest between stations in buffers. A transfer line with a reliable machine, an unreliable machine and a storage buffer is a typical example of such systems. This transfer line consists of two machines separated by a storage buffer. In this transfer line, it is assumed that workpieces (or parts) enter machine 1 from outside. Each workpiece is first operated upon in machine 1. After being processed by machine 1, a piece will directly pass to the buffer with probability η_1, otherwise it will be reprocessed by machine 1 with probability $q_1 = 1 - \eta_1$, which is the reprocessed rate of a piece on machine 1. From the buffer a piece will proceed to machine 2. After being operated on in machine 2, a piece will exit the system with probability η_2, or it will be reprocessed by machine 2 with probability $q_2 = 1 - \eta_2$. Machine 1 processes pieces according to a Poisson process with rate λ, and repair time of machine 1 follows an exponential distribution with rate β. Moreover, we assume that pieces exit from the system according to a Poisson process with rate λ. From the properties of a Poisson process we have

$$P\left\{\begin{matrix}\text{machine 1 transferred one}\\ \text{piece to the buffer during } \Delta t\end{matrix}\right\} = \lambda\eta_1\Delta t + o(\Delta t), \tag{4-1}$$

$$P\left\{\begin{matrix}\text{machine 1 transferred more than}\\ \text{one piece to the buffer during } \Delta t\end{matrix}\right\} = o(\Delta t), \tag{4-2}$$

$$P\left\{\begin{matrix}\text{a piece exits from}\\ \text{the system during } \Delta t\end{matrix}\right\} = \eta_2\Delta t + o(\Delta t), \tag{4-3}$$

$$P\left\{\begin{matrix}\text{more than one piece exit}\\ \text{from the system during } \Delta t\end{matrix}\right\} = o(\Delta t), \tag{4-4}$$

$$P\left\{\begin{matrix}\text{machine 2 reprocesses}\\ \text{a piece during } \Delta t\end{matrix}\right\} = q_2\Delta t + o(\Delta t). \tag{4-5}$$

The processing time of machine 2 has general continuous distribution

$$F(t) = 1 - e^{-\int_0^t \mu(x)dx}, \quad t \geq 0.$$

From this, together with the properties of exponential distribution and the Poisson process, we know

$$\begin{aligned} P&\begin{Bmatrix} \text{two machines do not complete} \\ \text{processes during } \Delta t \end{Bmatrix} \\ &= [1 - (\lambda\eta_1 + \mu(x))\Delta t] + o(\Delta t). \end{aligned} \tag{4-6}$$

From the definition and properties of $F(t)$ it is easy to see that $\mu(x)$, which is called the hazard rate, satisfies

$$\mu(x) \geq 0, \quad \int_0^\infty \mu(x)dx = \infty. \tag{4-7}$$

In addition, we assume that the storage buffer can store infinitely many materials. This is the number of workpieces in the buffer plus the number (0 or 1) of workpieces in machine 2. There is no workpiece in machine 2 if the buffer becomes empty and machine 2 completes the last remaining workpiece. Moreover, even if a machine is operative, it can not process workpieces if none are available to it or has 1 workpiece in the storage containing 1 on machine 2. In the former condition, the machine 2 is said to be starved, in the latter case machine 1 is called blocked. It is assumed that the first machine is never starved and the second machine is never blocked. We also assume machine 1 is reliable and each machine processes workpieces one by one. Let $Y(t)$ be the state at time t and $X(t)$ be the elapsed processing time of the piece currently being processed at time t; then $\{Y(t), X(t)\}$ forms a vector Markov process. According to the definition of the system, which has infinitely many states:

$Y(t) = 0$ represents that at time t, machine 1 is processing a piece and there are no pieces in the buffer and on the machine 2.

$Y(t) = n$ represents that at time t, two machines are both processing pieces and there are $n - 1$ pieces in the buffer, $n = 1, 2, \ldots$.

If we denote by

$$\begin{aligned} p_0(t) &= P\{Y(t) = 0\} \\ &= P\begin{Bmatrix} \text{at time } t \text{ machine 1 is processing} \\ \text{a piece and there are no pieces} \\ \text{in the buffer and on the machine 2} \end{Bmatrix}, \end{aligned} \tag{4-8}$$

$$
\begin{aligned}
p_n(x,t)dx &= P\{Y(t)=n, x \le X(t) < x+dx\} \\
&= P\left\{\begin{array}{c}\text{two machines are both}\\ \text{processing pieces and}\\ \text{there are } n-1 \text{ pieces}\\ \text{in the buffer, the elapsed}\\ \text{processing time of the}\\ \text{piece currently being}\\ \text{processed at time } t\\ \text{lies in } [x, x+dx)\end{array}\right\}, \quad n \ge 1, \qquad (4\text{-}9)
\end{aligned}
$$

then through considering that the transition occurred in Δt, and by using the total probability law, the Markov property and (4-1)–(4-6) we have (for convenience take the state transition Δx and the time transition Δt as the same Δt)

$$
\begin{aligned}
&p_0(t+\Delta t) = P\{Y(t+\Delta t)=0\} \\
&= P\left\{\begin{array}{c}\text{machine 1 is processing a piece}\\ \text{and there are no pieces in the}\\ \text{buffer and on the machine 2 at}\\ \text{time } t, \text{ machine 1 does not}\\ \text{complete the process during } \Delta t\end{array}\right\} \\
&\quad + P\left\{\begin{array}{c}\text{machine 1 is processing a piece,}\\ \text{machine 2 is processing a piece}\\ \text{and there are no pieces in the}\\ \text{buffer at time } t, \text{ one piece}\\ \text{exists from the system during } \Delta t\end{array}\right\} \\
&\quad + o(\Delta t) \\
&= P\left\{\begin{array}{c}\text{machine 1 is processing a piece}\\ \text{and there are no pieces in the}\\ \text{buffer and on the machine 2 at}\\ \text{time } t\end{array}\right\} \\
&\quad \times P\left\{\begin{array}{c}\text{machine 1 does not complete}\\ \text{the process during } \Delta t\end{array}\right\} \\
&\quad + P\left\{\begin{array}{c}\text{machine 1 is processing a piece,}\\ \text{machine 2 is processing a piece}\\ \text{and there are no pieces in the}\\ \text{buffer at time } t\end{array}\right\} \\
&\quad \times P\left\{\begin{array}{c}\text{one piece exists from}\\ \text{the system during } \Delta t\end{array}\right\} \\
&\quad + o(\Delta t) \\
&= p_0(t)(1-\lambda\eta_1\Delta t) + \int_0^\infty p_1(x,t)\mu(x)dx\eta_2\Delta t + o(\Delta t)
\end{aligned}
$$

$$\Longrightarrow$$

$$\begin{aligned}
&\frac{p_0(t+\Delta t)-p_0(t)}{\Delta t}\\
&=-\lambda\eta_1 p_0(t)+\eta_2\int_0^\infty p_1(x,t)\mu(x)dx+\frac{o(\Delta t)}{\Delta t}
\end{aligned}$$

$$\Longrightarrow$$

$$\begin{aligned}
&\lim_{\Delta t\to 0}\frac{p_0(t+\Delta t)-p_0(t)}{\Delta t}\\
&=-\lim_{\Delta t\to 0}\lambda\eta_1 p_0(t)+\lim_{\Delta t\to 0}\eta_2\int_0^\infty p_1(x,t)\mu(x)dx\\
&\quad+\lim_{\Delta t\to 0}\frac{o(\Delta t)}{\Delta t}
\end{aligned}$$

$$\Longrightarrow$$

$$\frac{dp_0(t)}{dt}=-\lambda\eta_1 p_0(t)+\eta_2\int_0^\infty \mu(x)p_1(x,t)dx. \tag{4-10}$$

$$\begin{aligned}
&p_1(x+\Delta x,t+\Delta t)\\
&=P\left\{\begin{matrix}\text{two machines are both processing}\\ \text{pieces and the elapsed processing}\\ \text{time of the piece currently being}\\ \text{processed at time } t \text{ lies in } [x,x+dx)\\ \text{and there are no pieces in the buffer}\\ \text{the processes are not completed and}\\ \text{there are no pieces in the buffer}\\ \text{during } \Delta t\end{matrix}\right\}\\
&\quad+o(\Delta t)\\
&=P\left\{\begin{matrix}\text{two machines are both processing pieces}\\ \text{and the elapsed processing time of}\\ \text{the piece currently being processed}\\ \text{at time } t \text{ lies in } [x,x+dx)\\ \text{and there are no pieces in the buffer}\end{matrix}\right\}\\
&\quad\times P\left\{\begin{matrix}\text{the processes are not completed and there}\\ \text{are no pieces in the buffer during } \Delta t\end{matrix}\right\}\\
&\quad+o(\Delta t)\\
&=p_1(x,t)[1-(\lambda\eta_1+\mu(x))\Delta t]+o(\Delta t)
\end{aligned}$$

$$\Longrightarrow$$

$$\begin{aligned}
p_1(x+\Delta t,t+\Delta t)-p_1(x,t)&=-(\lambda\eta_1+\mu(x))p_1(x,t)\Delta t\\
&\quad+o(\Delta t)
\end{aligned}$$

$$\Longrightarrow$$

$$\begin{aligned}
&\frac{p_1(x+\Delta t, t+\Delta t) - p_1(x+\Delta t, t)}{\Delta t} + \frac{p_1(x+\Delta t, t) - p_1(x,t)}{\Delta t} \\
&= -(\lambda\eta_1 + \mu(x))p_1(x,t) + \frac{o(\Delta t)}{\Delta t} \\
&\Longrightarrow \\
&\lim_{\Delta t\to 0} \frac{p_1(x+\Delta t, t+\Delta t) - p_1(x+\Delta t, t)}{\Delta t} \\
&\quad + \lim_{\Delta t\to 0} \frac{p_1(x+\Delta t, t) - p_1(x,t)}{\Delta t} \\
&= -\lim_{\Delta t\to 0}(\lambda\eta_1 + \mu(x))p_1(x,t) + \lim_{\Delta t\to 0}\frac{o(\Delta t)}{\Delta t} \\
&\Longrightarrow \\
&\frac{\partial p_1(x,t)}{\partial t} + \frac{\partial p_1(x,t)}{\partial x} = -(\lambda\eta_1 + \mu(x))p_1(x,t). \qquad (4\text{-}11)
\end{aligned}$$

$$\begin{aligned}
&p_n(x+\Delta x, t+\Delta t) \\
&= P\left\{\begin{matrix}\text{two machines are both processing pieces and} \\ \text{there are } n-1 \text{ pieces in the buffer and} \\ \text{the elapsed processing time of the piece} \\ \text{currently being processed at time } t \\ \text{lies in } [x, x+dx),\ \text{two machines do} \\ \text{not complete the processes during } \Delta t\end{matrix}\right\} \\
&\quad + P\left\{\begin{matrix}\text{two machines are both processing pieces} \\ \text{and there are } n-2 \text{ pieces in the buffer} \\ \text{and the elapsed processing time of the} \\ \text{piece currently being processed at time } t \\ \text{lies in } [x, x+dx),\ \text{machine 1 transferred} \\ \text{a piece to the buffer during } \Delta t\end{matrix}\right\} \\
&\quad + o(\Delta t) \\
&= P\left\{\begin{matrix}\text{two machines are both processing pieces} \\ \text{and there are } n-1 \text{ pieces in the} \\ \text{buffer and the elapsed processing time} \\ \text{of the piece currently being processed} \\ \text{at time } t \text{ lies in } [x, x+dx)\end{matrix}\right\} \\
&\quad \times P\left\{\begin{matrix}\text{two machines do not complete} \\ \text{the processes during } \Delta t\end{matrix}\right\} \\
&\quad + P\left\{\begin{matrix}\text{two machines are both processing pieces} \\ \text{and there are } n-2 \text{ pieces in the buffer} \\ \text{and the elapsed processing time of the} \\ \text{piece currently being processed at time} \\ t \text{ lies in } [x, x+dx)\end{matrix}\right\}
\end{aligned}$$

$$
\begin{aligned}
&\quad \times P \left\{ \begin{matrix} \text{machine 1 transferred a piece to} \\ \text{the buffer during } \Delta t \end{matrix} \right\} \\
&\quad + o(\Delta t) \\
&= p_n(x,t)[1 - (\lambda\eta_1 + \mu(x))\Delta t] + p_{n-1}(x,t)\lambda\eta_1 \Delta t + o(\Delta t) \\
&\Longrightarrow \\
&p_n(x + \Delta t, t + \Delta t) - p_n(x,t) \\
&= -(\lambda\eta_1 + \mu(x))p_n(x,t)\Delta t + \lambda\eta_1 \Delta t p_{n-1}(x,t) + o(\Delta t) \\
&\Longrightarrow \\
&\frac{p_n(x + \Delta t, t + \Delta t) - p_n(x + \Delta t, t)}{\Delta t} + \frac{p_n(x + \Delta t, t) - p_n(x,t)}{\Delta t} \\
&= -(\lambda\eta_1 + \mu(x))p_n(x,t) + \lambda\eta_1 p_{n-1}(x,t) + \frac{o(\Delta t)}{\Delta t} \\
&\Longrightarrow \\
&\lim_{\Delta t \to 0} \frac{p_n(x + \Delta t, t + \Delta t) - p_n(x + \Delta t, t)}{\Delta t} \\
&\quad + \lim_{\Delta t \to 0} \frac{p_n(x + \Delta t, t) - p_n(x,t)}{\Delta t} \\
&= -\lim_{\Delta t \to 0} (\lambda\eta_1 + \mu(x))p_n(x,t) + \lim_{\Delta t \to 0} \lambda\eta_1 p_{n-1}(x,t) \\
&\quad + \lim_{\Delta t \to 0} \frac{o(\Delta t)}{\Delta t} \\
&\Longrightarrow \\
&\frac{\partial p_n(x,t)}{\partial t} + \frac{\partial p_n(x,t)}{\partial x} \\
&= -(\lambda\eta_1 + \mu(x))p_n(x,t) + \lambda\eta_1 p_{n-1}(x,t), \quad n \geq 2.
\end{aligned} \tag{4-12}
$$

In the following we discuss boundary conditions and initial conditions.

$$
\begin{aligned}
p_1(0,t) &= P \left\{ \begin{matrix} \text{two machines are both processing} \\ \text{pieces and there are no pieces in} \\ \text{the buffer and on machine 2 at time } t \end{matrix} \right\} \\
&= P \left\{ \begin{matrix} \text{machine 1 has just processed a piece} \\ \text{there are no pieces in the buffer} \\ \text{and on machine 2 at time } t \end{matrix} \right\} \\
&\quad + P \left\{ \begin{matrix} \text{machine 2 just reprocessed one} \\ \text{piece and there are no pieces} \\ \text{in the buffer at time } t \end{matrix} \right\} \\
&\quad + P \left\{ \begin{matrix} \text{machine 2 just processed well one} \\ \text{piece and there are no pieces} \\ \text{in the buffer at time } t \end{matrix} \right\}
\end{aligned}
$$

$$
\begin{aligned}
&= \lambda\eta_1 p_0(t) + q_2 \int_0^\infty \mu(x) p_1(x,t)dx \\
&\quad + \eta_2 \int_0^\infty \mu(x) p_2(x,t)dx.
\end{aligned} \tag{4-13}
$$

$$
\begin{aligned}
p_n(0,t) &= P\left\{\begin{matrix}\text{two machines are both processing} \\ \text{pieces and there are } n-1 \text{ pieces} \\ \text{in the buffer at time } t\end{matrix}\right\} \\
&= P\left\{\begin{matrix}\text{two machines are both processing} \\ \text{pieces, there are } n-1 \text{ pieces in} \\ \text{the buffer and machine 2 just} \\ \text{reprocessed a piece at time } t\end{matrix}\right\} \\
&\quad + P\left\{\begin{matrix}\text{there are } n \text{ pieces in the} \\ \text{buffer and one piece just exited} \\ \text{from the system at time } t\end{matrix}\right\} \\
&= q_2 \int_0^\infty p_n(x,t)\mu(x)dx \\
&\quad + \eta_2 \int_0^\infty p_{n+1}(x,t)\mu(x)dx, \quad n \geq 2.
\end{aligned} \tag{4-14}
$$

If we assume that at time $t = 0$ there are no pieces on machine 1 and machine 2, then

$$
p_0(0) = 1, \quad p_n(x,0) = 0, \quad n \geq 1. \tag{4-15}
$$

Combining the above equations (4-10)–(4-15) we obtain the following mathematical model describing the above transfer line:

$$
\frac{dp_0(t)}{dt} = -\lambda\eta_1 p_0(t) + \eta_2 \int_0^\infty p_1(x,t)\mu(x)dx, \tag{4-16}
$$

$$
\frac{\partial p_1(x,t)}{\partial t} + \frac{\partial p_1(x,t)}{\partial x} = -(\lambda\eta_1 + \mu(x))p_1(x,t), \tag{4-17}
$$

$$
\begin{aligned}
\frac{\partial p_n(x,t)}{\partial t} + \frac{\partial p_n(x,t)}{\partial x} &= -(\lambda\eta_1 + \mu(x))p_n(x,t) \\
&\quad + \lambda\eta_1 p_{n-1}(x,t), \quad n \geq 2,
\end{aligned} \tag{4-18}
$$

$$
\begin{aligned}
p_1(0,t) &= \lambda\eta_1 p_0(t) + q_2 \int_0^\infty p_1(x,t)\mu(x)dx \\
&\quad + \eta_2 \int_0^\infty p_2(x,t)\mu(x)dx,
\end{aligned} \tag{4-19}
$$

$$
\begin{aligned}
p_n(0,t) &= q_2 \int_0^\infty p_n(x,t)\mu(x)dx \\
&\quad + \eta_2 \int_0^\infty p_{n+1}(x,t)\mu(x)dx, \quad n \geq 2,
\end{aligned} \tag{4-20}
$$

$$p_0(0) = 1, \quad p_n(x, 0) = 0, \quad n \geq 1. \tag{4-21}$$

Remark 4.1. From the definition of the transfer line we know that at $(x, t) = (0, 0)$ the system is empty, i.e., $\eta_1 = 0$, $\eta_2 = 0$ and therefore through inserting (4-21) into (4-19) and (4-20) it gives

$$p_1(0, 0) = 0, \quad p_n(0, 0) = 0, \quad n \geq 2, \tag{4-22}$$

which confirms (4-21). In other words, the system satisfies the compatibility condition.

For simplicity, we introduce a notation as follows.

$$\Gamma = \begin{pmatrix} e^{-x} & 0 & 0 & 0 & 0 & \cdots \\ \lambda\eta_1 e^{-x} & q_2\mu(x) & \eta_2\mu(x) & 0 & 0 & \cdots \\ 0 & 0 & q_2\mu(x) & \eta_2\mu(x) & 0 & \cdots \\ 0 & 0 & 0 & q_2\mu(x) & \eta_2\mu(x) & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

If we take a state space

$$X = \left\{ y \;\middle|\; \begin{array}{l} y \in \mathbb{R} \times L^1[0, \infty) \times L^1[0, \infty) \times \cdots, \\ \|y\| = |y_0| + \sum\limits_{n=1}^{\infty} \|y_n\|_{L^1[0,\infty)} < \infty \end{array} \right\},$$

then it is obvious that X is a Banach space. X is also a Banach lattice (see Definition 1.10). In fact, we define

$$\begin{aligned} &|y| = (|y_0|, |y_1(x)|, |y_2(x)|, |y_3(x)|, \ldots), \quad \forall y \in X, \\ &|y| \leq |p| \\ &\quad \Longleftrightarrow \\ &|y_0| \leq |p_0|, \ |y_n(x)| \leq |p_n(x)|, \quad n \geq 1, \ y, p \in X. \end{aligned}$$

Then $|y| \leq |p|, \ y, p \in X$ implies

$$\begin{aligned} &|y_0| \leq |p_0|, \ |y_n(x)| \leq |p_n(x)|, \ n \geq 1 \\ &\Longrightarrow \\ &|y_0| \leq |p_0|, \ \int_0^\infty |y_n(x)| dx \leq \int_0^\infty |p_n(x)| dx, \quad n \geq 1 \\ &\Longrightarrow \\ &|y_0| + \sum_{n=1}^{\infty} \int_0^\infty |y_n(x)| dx \leq |p_0| + \sum_{n=1}^{\infty} \int_0^\infty |p_n(x)| dx \\ &\Longrightarrow \\ &\|y\| \leq \|p\|. \end{aligned}$$

From which together with Definition 1.10 we know that X is a Banach lattice. In the following we define operators and their domains.

$$A\begin{pmatrix}p_0\\p_1\\p_2\\p_3\\\vdots\end{pmatrix}(x)=\begin{pmatrix}0&0&0&0&\cdots\\0&-\frac{d}{dx}&0&0&\cdots\\0&0&-\frac{d}{dx}&0&\cdots\\0&0&0&-\frac{d}{dx}&\cdots\\\vdots&\vdots&\vdots&\vdots&\ddots\end{pmatrix}\begin{pmatrix}p_0\\p_1(x)\\p_2(x)\\p_3(x)\\\vdots\end{pmatrix},$$

$$D(A)=\left\{y\in X\;\middle|\;\begin{array}{l}\frac{dy_n}{dx}\in L^1[0,\infty),\ y_n\ (n\geq 1)\text{ are}\\ \text{absolutely continuous functions}\\ \text{and satisfy } y(0)=\int_0^\infty \Gamma y(x)dx\end{array}\right\};$$

$$U\begin{pmatrix}p_0\\p_1\\p_2\\p_3\\\vdots\end{pmatrix}(x)=\begin{pmatrix}-\lambda\eta_1&0&0&0&\cdots\\0&-(\lambda\eta_1+\mu(x))&0&0&\cdots\\0&\lambda\eta_1&-(\lambda\eta_1+\mu(x))&0&\cdots\\0&0&\lambda\eta_1&-(\lambda\eta_1+\mu(x))&\cdots\\\vdots&\vdots&\vdots&\vdots&\ddots\end{pmatrix}\begin{pmatrix}p_0\\p_1(x)\\p_2(x)\\p_3(x)\\\vdots\end{pmatrix},\ D(U)=X;$$

$$E\begin{pmatrix}p_0\\p_1\\p_2\\\vdots\end{pmatrix}(x)=\begin{pmatrix}\eta_2\int_0^\infty p_1(x)\mu(x)dx\\0\\0\\\vdots\end{pmatrix},\quad D(E)=X,$$

then the above equations (4-16)–(4-21) can be written as an abstract Cauchy problem in the Banach space X :

$$\begin{cases}\frac{dp(t)}{dt}=(A+U+E)p(t), & t\in(0,\infty),\\ p(0)=(1,0,0,\ldots).\end{cases}\tag{4-23}$$

4.2 Well-posedness of the System (4-23)

Theorem 4.2. *If* $M = \sup_{x\in[0,\infty)} \mu(x) < \infty$*, then* $A + U + E$ *generates a positive contraction* C_0*-semigroup* $T(t)$*.*

Proof. We split the proof of this theorem into four steps. Firstly, we prove that $(\gamma I - A)^{-1}$ exists and is bounded for some γ. Secondly, we prove that $D(A)$ is dense in X. Thirdly, we verify that U and E are bounded linear operators. Thus by using the Hille-Yosida theorem and the perturbation theorem of a C_0-semigroup we deduce that $A + U + E$ generates a C_0-semigroup $T(t)$. Finally, we show that $A + U + E$ is a dispersive operator. Thus by the Phillips theorem we obtain that $T(t)$ is a positive contraction C_0-semigroup.

For any given $y \in X$ consider the equation $(\gamma I - A)p = y$. It is equivalent to

$$\gamma p_0 = y_0, \tag{4-24}$$

$$\frac{dp_n(x)}{dx} = -\gamma p_n(x) + y_n(x), \quad n \geq 1, \tag{4-25}$$

$$\begin{aligned} p_1(0) = \lambda\eta_1 p_0 + q_2 \int_0^\infty p_1(x)\mu(x)dx \\ + \eta_2 \int_0^\infty p_2(x)\mu(x)dx, \end{aligned} \tag{4-26}$$

$$\begin{aligned} p_n(0) = q_2 \int_0^\infty p_n(x)\mu(x)dx \\ + \eta_2 \int_0^\infty p_{n+1}(x)\mu(x)dx, \quad n \geq 2. \end{aligned} \tag{4-27}$$

Through solving (4-24) and (4-25) we have

$$p_0 = \frac{1}{\gamma} y_0, \tag{4-28}$$

$$p_n(x) = a_n e^{-\gamma x} + e^{-\gamma x} \int_0^x y_n(\tau)e^{\gamma\tau}d\tau, \quad n \geq 1. \tag{4-29}$$

By combining (4-29) with (4-26) and (4-27) we obtain

$$\begin{aligned} a_1 = p_1(0) &= \lambda\eta_1 p_0 \\ &+ q_2 \int_0^\infty \mu(x)\left[a_1 e^{-\gamma x} + e^{-\gamma x}\int_0^x y_1(\tau)e^{\gamma\tau}d\tau\right]dx \\ &+ \eta_2 \int_0^\infty \mu(x)\left[a_2 e^{-\gamma x} + e^{-\gamma x}\int_0^x y_2(\tau)e^{\gamma\tau}d\tau\right]dx \\ &= \lambda\eta_1 p_0 + a_1 q_2 \int_0^\infty \mu(x)e^{-\gamma x}dx \end{aligned}$$

$$
\begin{aligned}
&+ q_2 \int_0^\infty \mu(x) e^{-\gamma x} \int_0^x y_1(\tau) e^{\gamma\tau} d\tau dx \\
&+ a_2\eta_2 \int_0^\infty \mu(x) e^{-\gamma x} dx \\
&+ \eta_2 \int_0^\infty \mu(x) e^{-\gamma x} \int_0^x y_2(\tau) e^{\gamma\tau} d\tau dx \\
\Longrightarrow& \\
&\left(1 - q_2 \int_0^\infty \mu(x) e^{-\gamma x} dx\right) a_1 \\
&= a_2\eta_2 \int_0^\infty \mu(x) e^{-\gamma x} dx + \lambda\eta_1 p_0 \\
&\quad + q_2 \int_0^\infty \mu(x) e^{-\gamma x} \int_0^x y_1(\tau) e^{\gamma\tau} d\tau dx \\
&\quad + \eta_2 \int_0^\infty \mu(x) e^{-\gamma x} \int_0^x y_2(\tau) e^{\gamma\tau} d\tau dx,
\end{aligned} \tag{4-30}
$$

$$
\begin{aligned}
a_n &= p_n(0) \\
&= q_2 \int_0^\infty \mu(x) \left[a_n e^{-\gamma x} + e^{-\gamma x} \int_0^x y_n(\tau) e^{\gamma\tau} d\tau\right] dx \\
&\quad + \eta_2 \int_0^\infty \mu(x) \left[a_{n+1} e^{-\gamma x} + e^{-\gamma x} \int_0^x y_{n+1}(\tau) e^{\gamma\tau} d\tau\right] dx \\
&= a_n q_2 \int_0^\infty \mu(x) e^{-\gamma x} dx \\
&\quad + q_2 \int_0^\infty \mu(x) e^{-\gamma x} \int_0^x y_n(\tau) e^{\gamma\tau} d\tau dx \\
&\quad + a_{n+1}\eta_2 \int_0^\infty \mu(x) e^{-\gamma x} dx \\
&\quad + \eta_2 \int_0^\infty \mu(x) e^{-\gamma x} \int_0^x y_{n+1}(\tau) e^{\gamma\tau} d\tau dx \\
\Longrightarrow& \\
&\left(1 - q_2 \int_0^\infty \mu(x) e^{-\gamma x} dx\right) a_n \\
&= a_{n+1}\eta_2 \int_0^\infty \mu(x) e^{-\gamma x} dx \\
&\quad + q_2 \int_0^\infty \mu(x) e^{-\gamma x} \int_0^x y_n(\tau) e^{\gamma\tau} d\tau dx \\
&\quad + \eta_2 \int_0^\infty \mu(x) e^{-\gamma x} \int_0^x y_{n+1}(\tau) e^{\gamma\tau} d\tau dx, \quad n \geq 2.
\end{aligned} \tag{4-31}
$$

If we set

$$\mathcal{C} = \begin{pmatrix} 1-\overline{\alpha} & -\overline{\beta} & 0 & 0 & 0 & \cdots \\ 0 & 1-\overline{\alpha} & -\overline{\beta} & 0 & 0 & \cdots \\ 0 & 0 & 1-\overline{\alpha} & -\overline{\beta} & 0 & \cdots \\ 0 & 0 & 0 & 1-\overline{\alpha} & -\overline{\beta} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

$$\overrightarrow{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ \vdots \end{pmatrix}, \quad \overline{\alpha} = q_2 \int_0^\infty \mu(x) e^{-\gamma x} dx,$$

$$\overline{\beta} = \eta_2 \int_0^\infty \mu(x) e^{-\gamma x} dx,$$

then (4-30) and (4-31) are equivalent to

$$\mathcal{C}\overrightarrow{a} = \begin{pmatrix} \lambda\eta_1 p_0 + q_2 \int_0^\infty \mu(x) e^{-\gamma x} \int_0^x y_1(\tau) e^{\gamma\tau} d\tau dx + \eta_2 \int_0^\infty \mu(x) e^{-\gamma x} \int_0^x y_2(\tau) e^{\gamma\tau} d\tau dx \\ q_2 \int_0^\infty \mu(x) e^{-\gamma x} \int_0^x y_2(\tau) e^{\gamma\tau} d\tau dx + \eta_2 \int_0^\infty \mu(x) e^{-\gamma x} \int_0^x y_3(\tau) e^{\gamma\tau} d\tau dx \\ q_2 \int_0^\infty \mu(x) e^{-\gamma x} \int_0^x y_3(\tau) e^{\gamma\tau} d\tau dx + \eta_2 \int_0^\infty \mu(x) e^{-\gamma x} \int_0^x y_4(\tau) e^{\gamma\tau} d\tau dx \\ \vdots \end{pmatrix}. \tag{4-32}$$

It is easy to calculate

$$\mathcal{C}^{-1} = \begin{pmatrix} \frac{1}{1-\overline{\alpha}} & \frac{\overline{\beta}}{(1-\overline{\alpha})^2} & \frac{\overline{\beta}^2}{(1-\overline{\alpha})^3} & \frac{\overline{\beta}^3}{(1-\overline{\alpha})^4} & \cdots \\ 0 & \frac{1}{1-\overline{\alpha}} & \frac{\overline{\beta}}{(1-\overline{\alpha})^2} & \frac{\overline{\beta}^2}{(1-\overline{\alpha})^3} & \cdots \\ 0 & 0 & \frac{1}{1-\overline{\alpha}} & \frac{\overline{\beta}}{(1-\overline{\alpha})^2} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

From which together with (4-32) we deduce

$$\begin{aligned} a_1 = {} & \lambda\eta_1 p_0 \left(1 - q_2 \int_0^\infty \mu(x) e^{-\gamma x} dx\right)^{-1} \\ & + \sum_{k=1}^\infty \frac{\left(\eta_2 \int_0^\infty \mu(x) e^{-\gamma x} dx\right)^{k-1}}{\left(1 - q_2 \int_0^\infty \mu(x) e^{-\gamma x} dx\right)^k} \\ & \times q_2 \int_0^\infty \mu(x) e^{-\gamma x} \int_0^x y_k(\tau) e^{\gamma\tau} d\tau dx \end{aligned}$$

$$
\begin{aligned}
&+\sum_{k=1}^{\infty}\frac{\left(\eta_2\int_0^\infty \mu(x)e^{-\gamma x}dx\right)^{k-1}}{\left(1-q_2\int_0^\infty \mu(x)e^{-\gamma x}dx\right)^k}\\
&\times \eta_2\int_0^\infty \mu(x)e^{-\gamma x}\int_0^x y_{k+1}(\tau)e^{\gamma\tau}d\tau dx, && \text{(4-33)}
\end{aligned}
$$

$$
\begin{aligned}
a_n = &\sum_{k=1}^{\infty}\frac{\left(\eta_2\int_0^\infty \mu(x)e^{-\gamma x}dx\right)^{k-1}}{\left(1-q_2\int_0^\infty \mu(x)e^{-\gamma x}dx\right)^k}\\
&\times q_2\int_0^\infty \mu(x)e^{-\gamma x}\int_0^x y_{k+n-1}(\tau)e^{\gamma\tau}d\tau dx\\
&+\sum_{k=1}^{\infty}\frac{\left(\eta_2\int_0^\infty \mu(x)e^{-\gamma x}dx\right)^{k-1}}{\left(1-q_2\int_0^\infty \mu(x)e^{-\gamma x}dx\right)^k}\\
&\times \eta_2\int_0^\infty \mu(x)e^{-\gamma x}\int_0^x y_{k+n}(\tau)e^{\gamma\tau}d\tau dx, \quad n\geq 2. && \text{(4-34)}
\end{aligned}
$$

By using the Fubini theorem (see Theorem 1.49) we estimate (4-29) as follows (assume $\gamma > 0$).

$$
\begin{aligned}
\|p_n\|_{L^1[0,\infty)} &= \int_0^\infty \left| a_n e^{-\gamma x} + e^{-\gamma x}\int_0^x y_n(\tau)e^{\gamma\tau}d\tau\right| dx\\
&\leq \frac{1}{\gamma}|a_n| + \int_0^\infty e^{-\gamma x}\int_0^x |y_n(\tau)|e^{\gamma\tau}d\tau dx\\
&= \frac{1}{\gamma}|a_n| + \int_0^\infty |y_n(\tau)|e^{\gamma\tau}\int_\tau^\infty e^{-\gamma x}dxd\tau\\
&= \frac{1}{\gamma}|a_n| + \frac{1}{\gamma}\|y_n\|_{L^1[0,\infty)}, \quad n\geq 1. && \text{(4-35)}
\end{aligned}
$$

Through combining (4-28) with (4-33) and (4-34) and using the Fubini theorem (see Theorem 1.49) we have (assume $\gamma > Mq_2$)

$$
\begin{aligned}
|a_1| \leq\ &\lambda\eta_1|p_0|\left(1-q_2\int_0^\infty \mu(x)e^{-\gamma x}dx\right)^{-1}\\
&+\sum_{k=1}^{\infty}\frac{\left(\eta_2\int_0^\infty \mu(x)e^{-\gamma x}dx\right)^{k-1}}{\left(1-q_2\int_0^\infty \mu(x)e^{-\gamma x}dx\right)^k}\\
&\times q_2\int_0^\infty \mu(x)e^{-\gamma x}\int_0^x |y_k(\tau)|e^{\gamma\tau}d\tau dx\\
&+\sum_{k=1}^{\infty}\frac{\left(\eta_2\int_0^\infty \mu(x)e^{-\gamma x}dx\right)^{k-1}}{\left(1-q_2\int_0^\infty \mu(x)e^{-\gamma x}dx\right)^k}\\
&\times \eta_2\int_0^\infty \mu(x)e^{-\gamma x}\int_0^x |y_{k+1}(\tau)|e^{\gamma\tau}d\tau dx
\end{aligned}
$$

$$
\begin{aligned}
&\leq \frac{\lambda\eta_1|p_0|}{1 - Mq_2\int_0^\infty e^{-\gamma x}dx} \\
&\quad + \sum_{k=1}^\infty \frac{(M\eta_2\int_0^\infty e^{-\gamma x}dx)^{k-1}}{(1 - Mq_2\int_0^\infty e^{-\gamma x}dx)^k} \\
&\quad \times Mq_2\int_0^\infty e^{-\gamma x}\int_0^x |y_k(\tau)|e^{\gamma\tau}d\tau dx \\
&\quad + \sum_{k=1}^\infty \frac{(M\eta_2\int_0^\infty e^{-\gamma x}dx)^{k-1}}{(1 - Mq_2\int_0^\infty e^{-\gamma x}dx)^k} \\
&\quad \times M\eta_2\int_0^\infty e^{-\gamma x}\int_0^x |y_{k+1}(\tau)|e^{\gamma\tau}d\tau dx \\
&= \frac{\lambda\eta_1|p_0|}{1 - \frac{Mq_2}{\gamma}} \\
&\quad + \sum_{k=1}^\infty \frac{\left(\frac{M\eta_2}{\gamma}\right)^{k-1}}{\left(1 - \frac{Mq_2}{\gamma}\right)^k} Mq_2\int_0^\infty |y_k(\tau)|e^{\gamma\tau}\int_\tau^\infty e^{-\gamma x}dxd\tau \\
&\quad + \sum_{k=1}^\infty \frac{\left(\frac{M\eta_2}{\gamma}\right)^{k-1}}{\left(1 - \frac{Mq_2}{\gamma}\right)^k} M\eta_2\int_0^\infty |y_{k+1}(\tau)|e^{\gamma\tau}\int_\tau^\infty e^{-\gamma x}dxd\tau \\
&= \frac{\gamma\lambda\eta_1}{\gamma - Mq_2}|p_0| + \sum_{k=1}^\infty \frac{\gamma(M\eta_2)^{k-1}}{(\gamma - Mq_2)^k}\frac{Mq_2}{\gamma}\|y_k\|_{L^1[0,\infty)} \\
&\quad + \sum_{k=1}^\infty \frac{\gamma(M\eta_2)^{k-1}}{(\gamma - Mq_2)^k}\frac{M\eta_2}{\gamma}\|y_{k+1}\|_{L^1[0,\infty)} \\
&= \frac{\gamma\lambda\eta_1}{\gamma - Mq_2}|p_0| + \sum_{k=1}^\infty \frac{(M\eta_2)^{k-1}}{(\gamma - Mq_2)^k}Mq_2\|y_k\|_{L^1[0,\infty)} \\
&\quad + \sum_{k=1}^\infty \left(\frac{M\eta_2}{\gamma - Mq_2}\right)^k \|y_{k+1}\|_{L^1[0,\infty)} \\
&= \frac{\gamma\lambda\eta_1}{\gamma - Mq_2}\frac{1}{\gamma}|y_0| + \sum_{k=1}^\infty \frac{M^k\eta_2^{k-1}q_2}{(\gamma - Mq_2)^k}\|y_k\|_{L^1[0,\infty)} \\
&\quad + \sum_{k=1}^\infty \left(\frac{M\eta_2}{\gamma - Mq_2}\right)^k \|y_{k+1}\|_{L^1[0,\infty)} \\
&= \frac{\lambda\eta_1}{\gamma - Mq_2}|y_0| + \sum_{k=1}^\infty \frac{M^k\eta_2^{k-1}q_2}{(\gamma - Mq_2)^k}\|y_k\|_{L^1[0,\infty)}
\end{aligned}
$$

$$+\sum_{k=1}^{\infty}\left(\frac{M\eta_2}{\gamma-Mq_2}\right)^k\|y_{k+1}\|_{L^1[0,\infty)}, \tag{4-36}$$

$$\begin{aligned}
|a_n| \le{}& \sum_{k=1}^{\infty}\frac{\left(\eta_2\int_0^\infty\mu(x)e^{-\gamma x}dx\right)^{k-1}}{\left(1-q_2\int_0^\infty\mu(x)e^{-\gamma x}dx\right)^k}\\
&\times q_2\int_0^\infty\mu(x)e^{-\gamma x}\int_0^x|y_{k+n-1}(\tau)|e^{\gamma\tau}d\tau dx\\
&+\sum_{k=1}^{\infty}\frac{\left(\eta_2\int_0^\infty\mu(x)e^{-\gamma x}dx\right)^{k-1}}{\left(1-q_2\int_0^\infty\mu(x)e^{-\gamma x}dx\right)^k}\\
&\times \eta_2\int_0^\infty\mu(x)e^{-\gamma x}\int_0^x|y_{k+n}(\tau)|e^{\gamma\tau}d\tau dx\\
\le{}& \sum_{k=1}^{\infty}\frac{\left(M\eta_2\int_0^\infty e^{-\gamma x}dx\right)^{k-1}}{\left(1-Mq_2\int_0^\infty e^{-\gamma x}dx\right)^k}\\
&\times Mq_2\int_0^\infty e^{-\gamma x}\int_0^x|y_{k+n-1}(\tau)|e^{\gamma\tau}d\tau dx\\
&+\sum_{k=1}^{\infty}\frac{\left(M\eta_2\int_0^\infty e^{-\gamma x}dx\right)^{k-1}}{\left(1-Mq_2\int_0^\infty e^{-\gamma x}dx\right)^k}\\
&\times M\eta_2\int_0^\infty e^{-\gamma x}\int_0^x|y_{k+n}(\tau)|e^{\gamma\tau}d\tau dx\\
={}& \sum_{k=1}^{\infty}\frac{\gamma(M\eta_2)^{k-1}}{(\gamma-Mq_2)^k}Mq_2\int_0^\infty|y_{k+n-1}(\tau)|e^{\gamma\tau}\int_\tau^\infty e^{-\gamma x}dxd\tau\\
&+\sum_{k=1}^{\infty}\frac{\gamma(M\eta_2)^{k-1}}{(\gamma-Mq_2)^k}M\eta_2\int_0^\infty|y_{k+n}(\tau)|e^{\gamma\tau}\int_\tau^\infty e^{-\gamma x}dxd\tau\\
={}& \sum_{k=1}^{\infty}\frac{M^k\eta_2^{k-1}q_2}{(\gamma-Mq_2)^k}\|y_{k+n-1}\|_{L^1[0,\infty)}\\
&+\sum_{k=1}^{\infty}\left(\frac{M\eta_2}{\gamma-Mq_2}\right)^k\|y_{k+n}\|_{L^1[0,\infty)}, \quad n\ge 2.
\end{aligned} \tag{4-37}$$

In (4-36) and (4-37) we have used the following inequalities:

$$\begin{aligned}
&\int_0^\infty\mu(x)e^{-\gamma x}dx \le M\int_0^\infty e^{-\gamma x}dx = \frac{M}{\gamma}.\\
&1-q_2\int_0^\infty\mu(x)e^{-\gamma x}dx\\
&\quad\ge 1-Mq_2\int_0^\infty e^{-\gamma x}dx = 1-\frac{Mq_2}{\gamma}.
\end{aligned}$$

By inserting (4-36) and (4-37) into (4-35) and using (4-28) and $q_2 + \eta_2 = 1$ gives for $\gamma > \max\left\{\frac{M(Mq_2-\lambda\eta_1)}{M-\lambda\eta_1}, M\right\}$,

$$\begin{aligned}
\|p\| &= |p_0| + \sum_{n=1}^{\infty} \|p_n\|_{L^1[0,\infty)} \\
&\le \frac{1}{\gamma}|y_0| + \frac{1}{\gamma}\sum_{n=1}^{\infty}|a_n| + \frac{1}{\gamma}\sum_{n=1}^{\infty}\|y_n\|_{L^1[0,\infty)} \\
&\le \frac{1}{\gamma}|y_0| + \frac{\lambda\eta_1}{\gamma(\gamma - Mq_2)}|y_0| \\
&\quad + \sum_{k=1}^{\infty}\frac{M^k\eta_2^{k-1}q_2}{\gamma(\gamma-Mq_2)^k}\|y_k\|_{L^1[0,\infty)} \\
&\quad + \sum_{k=1}^{\infty}\frac{1}{\gamma}\left(\frac{M\eta_2}{\gamma-Mq_2}\right)^k\|y_{k+1}\|_{L^1[0,\infty)} \\
&\quad + \sum_{n=2}^{\infty}\sum_{k=1}^{\infty}\frac{M^k\eta_2^{k-1}q_2}{\gamma(\gamma-Mq_2)^k}\|y_{k+n-1}\|_{L^1[0,\infty)} \\
&\quad + \sum_{n=2}^{\infty}\sum_{k=1}^{\infty}\left(\frac{M\eta_2}{\gamma-Mq_2}\right)^k\frac{1}{\gamma}\|y_{k+n}\|_{L^1[0,\infty)} \\
&\quad + \frac{1}{\gamma}\sum_{n=1}^{\infty}\|y_n\|_{L^1[0,\infty)} \\
&= \frac{\gamma+\lambda\eta_1-Mq_2}{\gamma(\gamma-Mq_2)}|y_0| \\
&\quad + \sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\frac{M^k\eta_2^{k-1}}{(\gamma-Mq_2)^k}\frac{q_2}{\gamma}\|y_{k+n-1}\|_{L^1[0,\infty)} \\
&\quad + \sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\left(\frac{M\eta_2}{\gamma-Mq_2}\right)^k\frac{1}{\gamma}\|y_{k+n}\|_{L^1[0,\infty)} \\
&\quad + \frac{1}{\gamma}\sum_{n=1}^{\infty}\|y_n\|_{L^1[0,\infty)} \\
&= \frac{\gamma+\lambda\eta_1-Mq_2}{\gamma(\gamma-Mq_2)}|y_0| \\
&\quad + \frac{q_2}{\gamma}\sum_{k=1}^{\infty}\frac{M^k\eta_2^{k-1}}{(\gamma-Mq_2)^k}\sum_{n=1}^{\infty}\|y_{k+n-1}\|_{L^1[0,\infty)} \\
&\quad + \frac{1}{\gamma}\sum_{k=1}^{\infty}\left(\frac{M\eta_2}{\gamma-Mq_2}\right)^k\sum_{n=1}^{\infty}\|y_{k+n}\|_{L^1[0,\infty)}
\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{\gamma}\sum_{n=1}^{\infty}\|y_n\|_{L^1[0,\infty)}\\
\le\ &\frac{\gamma+\lambda\eta_1-Mq_2}{\gamma(\gamma-Mq_2)}|y_0|\\
&+\frac{q_2}{\gamma}\sum_{k=1}^{\infty}\frac{M^k\eta_2^{k-1}}{(\gamma-Mq_2)^k}\sum_{n=1}^{\infty}\|y_n\|_{L^1[0,\infty)}\\
&+\frac{1}{\gamma}\sum_{k=1}^{\infty}\left(\frac{M\eta_2}{\gamma-Mq_2}\right)^k\sum_{n=1}^{\infty}\|y_n\|_{L^1[0,\infty)}\\
&+\frac{1}{\gamma}\sum_{n=1}^{\infty}\|y_n\|_{L^1[0,\infty)}\\
=\ &\frac{\gamma+\lambda\eta_1-Mq_2}{\gamma(\gamma-Mq_2)}|y_0|\\
&+\frac{q_2}{\gamma}\frac{M}{\gamma-Mq_2-M\eta_2}\sum_{n=1}^{\infty}\|y_n\|_{L^1[0,\infty)}\\
&+\frac{1}{\gamma}\frac{M\eta_2}{\gamma-Mq_2-M\eta_2}\sum_{n=1}^{\infty}\|y_n\|_{L^1[0,\infty)}\\
&+\frac{1}{\gamma}\sum_{n=1}^{\infty}\|y_n\|_{L^1[0,\infty)}\\
=\ &\frac{\gamma+\lambda\eta_1-Mq_2}{\gamma(\gamma-Mq_2)}|y_0|\\
&+\left(\frac{Mq_2}{\gamma(\gamma-Mq_2-M\eta_2)}+\frac{M\eta_2}{\gamma(\gamma-Mq_2-M\eta_2)}\right.\\
&\qquad\left.+\frac{1}{\gamma}\right)\sum_{n=1}^{\infty}\|y_n\|_{L^1[0,\infty)}\\
=\ &\frac{\gamma+\lambda\eta_1-Mq_2}{\gamma(\gamma-Mq_2)}|y_0|+\frac{1}{\gamma-Mq_2-M\eta_2}\sum_{n=1}^{\infty}\|y_n\|_{L^1[0,\infty)}\\
\le\ &\frac{1}{\gamma-Mq_2-M\eta_2}\left(|y_0|+\sum_{n=1}^{\infty}\|y_n\|_{L^1[0,\infty)}\right)\\
=\ &\frac{1}{\gamma-M(q_2+\eta_2)}\|y\|\\
=\ &\frac{1}{\gamma-M}\|y\|. \qquad (4\text{-}38)
\end{aligned}
$$

In (4-38) we have applied the following inequality:

$$
\begin{aligned}
&\gamma > \frac{M(Mq_2 - \lambda\eta_1)}{M - \lambda\eta_1} \\
&\Longrightarrow \\
&\gamma(M - \lambda\eta_1) > M(Mq_2 - \lambda\eta_1) \\
&\Longrightarrow \\
&\gamma(M - \lambda\eta_1) > (Mq_2 - \lambda\eta_1)(Mq_2 + M\eta_2) \\
&\Longrightarrow \\
&\gamma(Mq_2 + M\eta_2 - \lambda\eta_1 + Mq_2 - Mq_2) \\
&\ > -(\lambda\eta_1 - Mq_2)(Mq_2 + M\eta_2) \\
&\Longrightarrow \\
&\gamma^2 - \gamma Mq_2 \\
&\ > \gamma^2 - \gamma(Mq_2 + M\eta_2) + \gamma(\lambda\eta_1 - Mq_2) \\
&\quad - (\lambda\eta_1 - Mq_2)(Mq_2 + M\eta_2) \\
&\Longrightarrow \\
&\gamma(\gamma - Mq_2) > (\gamma + \lambda\eta_1 - Mq_2)(\gamma - Mq_2 - M\eta_2) \\
&\Longrightarrow \\
&\frac{\gamma + \lambda\eta_1 - Mq_2}{\gamma(\gamma - Mq_2)} < \frac{1}{\gamma - Mq_2 - M\eta_2}.
\end{aligned}
$$

(4-38) shows that when $\gamma > \max\left\{\frac{M(Mq_2-\lambda\eta_1)}{M-\lambda\eta_1}, M\right\}$,

$$(\gamma I - A)^{-1} : X \to D(A), \quad \left\|(\gamma I - A)^{-1}\right\| \le \frac{1}{\gamma - M}.$$

As far as the second step is concerned, from $|p_0| + \sum_{n=1}^{\infty} \|p_n\|_{L^1[0,\infty)} < \infty$ for $p \in X$ it follows that, for any $\epsilon > 0$, there is a positive integer N such that $\sum_{n=N}^{\infty} \|p_n\|_{L^1[0,\infty)} < \epsilon$. Let

$$\mathcal{L} = \left\{ (p_0, p_1, p_2, \ldots, p_N, 0, 0, \ldots) \;\middle|\; \begin{array}{l} p_i \in L^1[0,\infty), \\ i = 1, 2, \ldots, N, \\ N \text{ is a finite} \\ \text{positive integer} \end{array} \right\}.$$

Here, $p \in \mathcal{L}$ means that the prior finite components of p are nonzero while the others are equal to zero. It is obvious that $\mathcal{L}$ is dense in X. If we set

$$Z = \left\{ (p_0, p_1, p_2, \dots, p_l, 0, 0, \dots) \;\middle|\; \begin{array}{l} p_i \in C_0^\infty[0,\infty), \text{ there} \\ \text{exist numbers } c_i > 0 \\ \text{such that } p_i(x) = 0 \text{ for} \\ x \in [0, c_i],\ i = 1, 2, \dots, l \end{array} \right\},$$

then by Adams [2] it is easy to show that Z is dense in $\mathcal{L}$.

From the above discussion we know that in order to prove $D(A)$ is dense in X, it is sufficient to prove $Z \subset \overline{D(A)}$. In fact, if $Z \subset \overline{D(A)}$, then $\overline{Z} \subset \overline{\overline{D(A)}} = \overline{D(A)}$, which together with $\overline{Z} = X$ we know that $X \subset \overline{D(A)}$. On the other hand, $D(A) \subset X$ implies $\overline{D(A)} \subset \overline{X} = X$ and therefore $\overline{D(A)} = X$.

Take any $p \in Z$, there exist a finite positive integer l and $c_i > 0$ $(i = 1, 2, \dots, l)$ such that

$$\begin{aligned} p(x) &= (p_0, p_1(x), p_2(x), \dots, p_l(x), 0, 0, \dots), \\ p_i(x) &= 0 \text{ for } \forall x \in [0, c_i], \quad i = 1, 2, \dots, l. \end{aligned}$$

Which leads to $p_i(x) = 0$ for $x \in [0, 2s]$, where $0 < 2s < \min\{c_1, c_2, \dots, c_l\}$. Define

$$\begin{aligned} f^s(0) &= (p_0, f_1^s(0), f_2^s(0), f_3^s(0), \dots, f_l^s(0), 0, 0, \dots) \\ &= (p_0, \lambda\eta_1 p_0 + q_2 \int_{2s}^\infty p_1(x)\mu(x)dx + \eta_2 \int_{2s}^\infty p_2(x)\mu(x)dx, \\ &\quad q_2 \int_{2s}^\infty p_2(x)\mu(x)dx + \eta_2 \int_{2s}^\infty p_3(x)\mu(x)dx, \dots, \\ &\quad q_2 \int_{2s}^\infty p_{l-1}(x)\mu(x)dx + \eta_2 \int_{2s}^\infty p_l(x)\mu(x)dx, \\ &\quad q_2 \int_{2s}^\infty p_l(x)\mu(x)dx, 0, 0, \dots) \\ f^s(x) &= (p_0, f_1^s(x), f_2^s(x), f_3^s(x), \dots, f_l^s(x), 0, 0, \dots), \end{aligned}$$

where

$$f_i^s(x) = \begin{cases} f_i^s(0)\left(1 - \frac{x}{s}\right)^2 & x \in [0, s), \\ -u_i(x-s)^2(x-2s)^2 & x \in [s, 2s), \\ p_i(x) & x \in [2s, \infty), \end{cases}$$

$$u_i = \frac{f_i^s(0)\int_0^s \mu(x)(1-\frac{x}{s})^2 dx}{\int_s^{2s} \mu(x)(x-s)^2(x-2s)^2 dx}, \quad i = 1, 2, \dots, l,$$

then it is not difficult to verify that $f^s \in D(A)$. Moreover,

$$\|p - f^s\| = \sum_{n=1}^{l} \int_0^\infty |p_n(x) - f_n^s(x)| dx$$

$$
\begin{aligned}
&= \sum_{n=1}^{l} \left[\int_0^s |f_n^s(x)|dx + \int_s^{2s} |f_n^s(x)|dx \right] \\
&= \sum_{n=1}^{l} |f_n^s(0)| \int_0^s \left(1 - \frac{x}{s}\right)^2 dx \\
&\quad + \sum_{n=1}^{l} |u_i| \int_s^{2s} (x-s)^2(x-2s)^2 dx \\
&= \sum_{n=1}^{l} |f_n^s(0)| \frac{s}{3} + \sum_{n=1}^{l} |u_i| \frac{s^5}{30} \\
&\to 0, \quad \text{as} \quad s \to 0,
\end{aligned}
$$

which shows $Z \subset \overline{D(A)}$. Hence, $D(A)$ is dense in X.

From the first step, the second step and the Hille-Yosida theorem (see Theorem 1.68) we know that A generates a C_0-semigroup.

In the following we will verify that U and E are bounded linear operators.

$$
Up(x) = \begin{pmatrix} -\lambda\eta_1 p_0 \\ -(\lambda\eta_1 + \mu(x))p_1(x) \\ -(\lambda\eta_1 + \mu(x))p_2(x) + \lambda\eta_1 p_1(x) \\ -(\lambda\eta_1 + \mu(x))p_3(x) + \lambda\eta_1 p_2(x) \\ \vdots \end{pmatrix}
$$

$$\Longrightarrow$$

$$
\begin{aligned}
\|Up\| &= |-\lambda\eta_1 p_0| + \int_0^\infty |-(\lambda\eta_1 + \mu(x))p_1(x)|dx \\
&\quad + \int_0^\infty |-(\lambda\eta_1 + \mu(x))p_2(x) + \lambda\eta_1 p_1(x)|dx \\
&\quad + \int_0^\infty |-(\lambda\eta_1 + \mu(x))p_3(x) + \lambda\eta_1 p_2(x)|dx \\
&\quad + \cdots \\
&\le \lambda\eta_1 |p_0| + \sum_{n=1}^{\infty} \int_0^\infty (\lambda\eta_1 + \mu(x))|p_n(x)|dx \\
&\quad + \sum_{n=1}^{\infty} \int_0^\infty \lambda\eta_1 |p_n(x)|dx \\
&\le \lambda\eta_1 |p_0| + \sum_{n=1}^{\infty} (\lambda\eta_1 + M) \int_0^\infty |p_n(x)|dx \\
&\quad + \lambda\eta_1 \sum_{n=1}^{\infty} \int_0^\infty |p_n(x)|dx
\end{aligned}
$$

$$
\begin{aligned}
&= \lambda\eta_1|p_0| + (\lambda\eta_1 + M)\sum_{n=1}^{\infty}\|p_n\|_{L^1[0,\infty)} \\
&\quad + \lambda\eta_1\sum_{n=1}^{\infty}\|p_n\|_{L^1[0,\infty)} \\
&\le (2\lambda\eta_1 + M)\left(|p_0| + \sum_{n=1}^{\infty}\|p_n\|_{L^1[0,\infty)}\right) \\
&= (2\lambda\eta_1 + M)\|p\|.
\end{aligned} \tag{4-39}
$$

$$
Ep(x) = \begin{pmatrix} \eta_2\int_0^{\infty}\mu(x)p_1(x)dx \\ 0 \\ 0 \\ \vdots \end{pmatrix}
$$

$$
\Longrightarrow
$$

$$
\begin{aligned}
\|Ep\| &= \left|\eta_2\int_0^{\infty}\mu(x)p_1(x)dx\right| + \int_0^{\infty}|0|dx + \int_0^{\infty}|0|dx + \cdots \\
&\le \eta_2\int_0^{\infty}\mu(x)|p_1(x)|dx \le M\eta_2\int_0^{\infty}|p_1(x)|dx \\
&= M\eta_2\|p_1\|_{L^1[0,\infty)} \le M\eta_2\|p\|.
\end{aligned} \tag{4-40}
$$

(4-39) and (4-40) show that U and E are bounded operators. It is easy to check that $U(\tilde{\alpha}p + \tilde{\beta}q) = \tilde{\alpha}Up + \tilde{\beta}Uq$ and $E(\tilde{\alpha}p + \tilde{\beta}q) = \tilde{\alpha}Ep + \tilde{\beta}Eq$ for all $\tilde{\alpha},\ \tilde{\beta} \in \mathbb{C}$ and $p,\ q \in X$. In other words, U and E are linear operators.

From the above step together with the first step, second step and the perturbation theorem of a C_0-semigroup (see Theorem 1.80) we obtain that $A+U+E$ generates a C_0-semigroup $T(t)$.

Lastly, we will prove that $A+U+E$ is a dispersive operator. For $p = (p_0, p_1, p_2, \dots) \in X$ we may choose

$$
\phi(x) = \left(\frac{[p_0]^+}{p_0}, \frac{[p_1(x)]^+}{p_1(x)}, \frac{[p_2(x)]^+}{p_2(x)}, \frac{[p_3(x)]^+}{p_3(x)}, \dots\right),
$$

where

$$
[p_0]^+ = \begin{cases} p_0 & \text{if } \ p_0 > 0, \\ 0 & \text{if } \ p_0 \le 0, \end{cases}
$$

$$
[p_i(x)]^+ = \begin{cases} p_i(x) & \text{if } \ p_i(x) > 0 \\ 0 & \text{if } \ p_i(x) \le 0 \end{cases}, \quad i = 1, 2, 3, \dots.
$$

The boundary conditions on $p \in D(A)$ imply

$$\begin{aligned}
\sum_{n=1}^{\infty}[p_n(0)]^+ &= [p_1(0)]^+ + \sum_{n=2}^{\infty}[p_n(0)]^+ \\
&\le \lambda\eta_1[p_0]^+ + q_2\int_0^\infty [p_1(x)]^+\mu(x)dx \\
&\quad + \eta_2\int_0^\infty [p_2(x)]^+\mu(x)dx \\
&\quad + \sum_{n=2}^{\infty} q_2\int_0^\infty [p_n(x)]^+\mu(x)dx \\
&\quad + \sum_{n=2}^{\infty} \eta_2\int_0^\infty [p_{n+1}(x)]^+\mu(x)dx \\
&= \lambda\eta_1[p_0]^+ + q_2\sum_{n=1}^{\infty}\int_0^\infty [p_n(x)]^+\mu(x)dx \\
&\quad + \sum_{n=1}^{\infty} \eta_2\int_0^\infty [p_{n+1}(x)]^+\mu(x)dx \\
&= \lambda\eta_1[p_0]^+ + (q_2+\eta_2)\sum_{n=2}^{\infty}\int_0^\infty [p_n(x)]^+\mu(x)dx \\
&\quad + q_2\int_0^\infty [p_1(x)]^+\mu(x)dx \\
&= \lambda\eta_1[p_0]^+ + \sum_{n=2}^{\infty}\int_0^\infty [p_n(x)]^+\mu(x)dx \\
&\quad + q_2\int_0^\infty [p_1(x)]^+\mu(x)dx. \qquad (4\text{-}41)
\end{aligned}$$

If we take $V_n = \{x \in [0,\infty) \mid p_n(x) > 0\}$ and $W_n = \{x \in [0,\infty) \mid p_n(x) \le 0\}$ for $n = 1, 2, 3, \dots$, then by a short argument we have, for $p \in D(A)$,

$$\begin{aligned}
&\int_0^\infty \frac{dp_n(x)}{dx}\frac{[p_n(x)]^+}{p_n(x)}dx \\
&= \int_{V_n} \frac{dp_n(x)}{dx}\frac{[p_n(x)]^+}{p_n(x)}dx + \int_{W_n} \frac{dp_n(x)}{dx}\frac{[p_n(x)]^+}{p_n(x)}dx \\
&= \int_{V_n} \frac{dp_n(x)}{dx}\frac{[p_n(x)]^+}{p_n(x)}dx = \int_{V_n}\frac{dp_n(x)}{dx}dx \\
&= \int_0^\infty \frac{d[p_n(x)]^+}{dx}dx = [p_n(x)]^+\Big|_0^\infty \\
&= -[p_n(0)]^+, \quad n \ge 1. \qquad (4\text{-}42)
\end{aligned}$$

By using the boundary conditions on $p \in D(A)$, (4-41), (4-42) and $q_2 + \eta_2 = 1$ it follows that

$$\begin{aligned}
&\langle (A+U+E)p, \phi\rangle \\
&= \left[-\lambda\eta_1 p_0 + \eta_2 \int_0^\infty p_1(x)\mu(x)dx\right] \frac{[p_0]^+}{p_0} \\
&\quad + \int_0^\infty \left[-\frac{dp_1(x)}{dx} - (\lambda\eta_1 + \mu(x))p_1(x)\right] \frac{[p_1(x)]^+}{p_1(x)} dx \\
&\quad + \sum_{n=2}^\infty \int_0^\infty \left[-\frac{dp_n(x)}{dx} - (\lambda\eta_1 + \mu(x))p_n(x)\right. \\
&\qquad \left. + \lambda\eta_1 p_{n-1}(x)\right] \frac{[p_n(x)]^+}{p_n(x)} dx \\
&= -\lambda\eta_1 [p_0]^+ + \eta_2 \frac{[p_0]^+}{p_0} \int_0^\infty p_1(x)\mu(x)dx \\
&\quad - \sum_{n=1}^\infty \int_0^\infty \left[\frac{dp_n(x)}{dx} + (\lambda\eta_1 + \mu(x))p_n(x)\right] \frac{[p_n(x)]^+}{p_n(x)} dx \\
&\quad + \lambda\eta_1 \sum_{n=2}^\infty \int_0^\infty p_{n-1}(x) \frac{[p_n(x)]^+}{p_n(x)} dx \\
&= -\lambda\eta_1 [p_0]^+ + \eta_2 \frac{[p_0]^+}{p_0} \int_0^\infty p_1(x)\mu(x)dx \\
&\quad - \sum_{n=1}^\infty \int_0^\infty \frac{dp_n(x)}{dx} \frac{[p_n(x)]^+}{p_n(x)} dx \\
&\quad - \sum_{n=1}^\infty \int_0^\infty (\lambda\eta_1 + \mu(x))[p_n(x)]^+ dx \\
&\quad + \lambda\eta_1 \sum_{n=2}^\infty \int_0^\infty p_{n-1}(x) \frac{[p_n(x)]^+}{p_n(x)} dx \\
&\le -\lambda\eta_1 [p_0]^+ + \eta_2 \frac{[p_0]^+}{p_0} \int_0^\infty p_1(x)\mu(x)dx \\
&\quad + \lambda\eta_1 [p_0]^+ + \sum_{n=2}^\infty \int_0^\infty [p_n(x)]^+ \mu(x)dx \\
&\quad + q_2 \int_0^\infty [p_1(x)]^+ \mu(x)dx \\
&\quad - \lambda\eta_1 \sum_{n=1}^\infty \int_0^\infty [p_n(x)]^+ dx - \sum_{n=1}^\infty \int_0^\infty [p_n(x)]^+ \mu(x)dx
\end{aligned}$$

$$
\begin{aligned}
&\quad + \lambda\eta_1 \sum_{n=2}^{\infty} \int_0^{\infty} p_{n-1}(x) \frac{[p_n(x)]^+}{p_n(x)} dx \\
&= \eta_2 \frac{[p_0]^+}{p_0} \int_0^{\infty} p_1(x)\mu(x)dx + q_2 \int_0^{\infty} [p_1(x)]^+ \mu(x)dx \\
&- \lambda\eta_1 \sum_{n=1}^{\infty} \int_0^{\infty} [p_n(x)]^+ dx - \int_0^{\infty} [p_1(x)]^+ \mu(x)dx \\
&+ \lambda\eta_1 \sum_{n=2}^{\infty} \int_0^{\infty} p_{n-1}(x) \frac{[p_n(x)]^+}{p_n(x)} dx \\
&\le \eta_2 \frac{[p_0]^+}{p_0} \int_0^{\infty} p_1(x)\mu(x)dx + q_2 \int_0^{\infty} [p_1(x)]^+ \mu(x)dx \\
&- \lambda\eta_1 \sum_{n=1}^{\infty} \int_0^{\infty} [p_n(x)]^+ dx - \int_0^{\infty} [p_1(x)]^+ \mu(x)dx \\
&+ \lambda\eta_1 \sum_{n=2}^{\infty} \int_0^{\infty} [p_{n-1}(x)]^+ dx \\
&= \eta_2 \frac{[p_0]^+}{p_0} \int_0^{\infty} p_1(x)\mu(x)dx \\
&\quad + q_2 \int_0^{\infty} [p_1(x)]^+ \mu(x)dx - \int_0^{\infty} [p_1(x)]^+ \mu(x)dx \\
&\le \eta_2 \frac{[p_0]^+}{p_0} \int_0^{\infty} [p_1(x)]^+ \mu(x)dx \\
&\quad + q_2 \int_0^{\infty} [p_1(x)]^+ \mu(x)dx - \int_0^{\infty} [p_1(x)]^+ \mu(x)dx \\
&= \left(\eta_2 \frac{[p_0]^+}{p_0} + q_2 - 1 \right) \int_0^{\infty} [p_1(x)]^+ \mu(x)dx \\
&= \left(\eta_2 \frac{[p_0]^+}{p_0} - \eta_2 \right) \int_0^{\infty} [p_1(x)]^+ \mu(x)dx \\
&\le 0.
\end{aligned} \tag{4-43}
$$

In (4-43) we have used the inequalities:

$$
\int_0^{\infty} p_1(x)\mu(x)dx \le \int_0^{\infty} [p_1(x)]^+ \mu(x)dx.
$$

$$
\begin{aligned}
\int_0^{\infty} p_{n-1}(x)dx \frac{[p_n(x)]^+}{p_n(x)} dx &\le \int_0^{\infty} [p_{n-1}(x)]^+ \frac{[p_n(x)]^+}{p_n(x)} dx \\
&\le \int_0^{\infty} [p_{n-1}(x)]^+ dx, \quad n \ge 2.
\end{aligned}
$$

From (4-43) together with the definition of a dispersive operator (see Definition 1.74 and Theorem 1.79) we know that $(A+U+E)$ is a dispersive operator. From which together with the first step, the second step and Theorem 1.79 we deduce that $A+U+E$ generates a positive contraction C_0-semigroup. By the uniqueness of a C_0-semigroup (see Remark 1.66) we conclude that this positive contraction C_0-semigroup is just $T(t)$. $\square$

Theorem 4.3. *If* $M = \sup\limits_{x\in[0,\infty)} \mu(x) < \infty$, *the system* (4-23) *has a unique nonnegative time-dependent solution* $p(x,t)$ *which satisfies*

$$\|p(\cdot,t)\| \leq 1, \quad \forall t \in [0,\infty). \tag{4-44}$$

Proof. Since the initial value $p(0) \in D(A)$, by Remark 4.1, Theorem 1.81 and Theorem 4.2 we know that the system (4-23) has a unique nonnegative time-dependent solution which can be expressed as

$$p(x,t) = T(t)(1,0,0,\dots), \quad \forall t \in [0,\infty). \tag{4-45}$$

Theorem 4.2 implies

$$\|T(t)\| \leq 1, \quad \forall t \in [0,\infty). \tag{4-46}$$

By combining (4-45) with (4-46) we have

$$\begin{aligned}\|p(\cdot,t)\| &= \|T(t)(1,0,0,\dots)\| \leq \|T(t)\|\|(1,0,0,\dots)\| \\ &\leq \|(1,0,0,\dots)\| = 1, \quad \forall t \in [0,\infty).\end{aligned}$$ $\square$

(4-44) shows that $p(x,t)$ conforms to its physical significance.

4.3 Asymptotic Behavior of the Time-dependent Solution of the System (4-23) When $\mu(x) = \mu$

In this section, by using the idea in Gupur et al. [65] we will study the asymptotic behavior of the time-dependent solution of the system (4-23) when the hazard rate $\mu(x)$ is a constant μ, that is, $\mu(x) = \mu$. We first determine $(A+U+E)^*$, the adjoint operator of $A+U+E$, then consider the resolvent set of $(A+U+E)^*$, next prove that 0 is an eigenvalue of $A+U+E$ and $(A+U+E)^*$ with geometric multiplicity one, and algebraic multiplicity of 0 in X^* is one. Thus we obtain that the time-dependent solution of the system (4-23) converges strongly to its steady-state solution. Lastly, by studying eigenvalues of $A+U+E$ on the left half complex plane we show that our result is the best result on convergence.

It is easy to check that X^*, dual space of X, is

$$X^* = \left\{ q^* \;\middle|\; \begin{array}{l} q^* \in \mathbb{R} \times L^\infty[0,\infty) \times L^\infty[0,\infty) \times \cdots, \\ |||q^*||| = \max\left\{ |q_0^*|, \sup\limits_{n\geq 1} \|q_n^*\|_{L^\infty[0,\infty)} \right\} < \infty \end{array} \right\}.$$

It is obvious that X^* is a Banach space.

Lemma 4.4. *$(A+U+E)^*$, the adjoint operator of $A+U+E$, is as follows:*

$$(A+U+E)^*q^* = (G+F+\mathbb{V})q^*, \quad \forall q^* \in D(G),$$

where

$$G\begin{pmatrix} q_0^* \\ q_1^* \\ q_2^* \\ \vdots \end{pmatrix}(x) = \begin{pmatrix} -\lambda\eta_1 & 0 & 0 & \cdots \\ 0 & \frac{d}{dx} - (\lambda\eta_1 + \mu(x)) & 0 & \cdots \\ 0 & 0 & \frac{d}{dx} - (\lambda\eta_1 + \mu(x)) & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} q_0^* \\ q_1^*(x) \\ q_2^*(x) \\ \vdots \end{pmatrix},$$

$$D(G) = \left\{ q^* \in X^* \;\middle|\; \begin{array}{l} \frac{dq_i^*(x)}{dx} \text{ exist and} \\ q_i^*(\infty) = \alpha, \quad n = 1, 2, \ldots \end{array} \right\};$$

$$F\begin{pmatrix} q_0^* \\ q_1^* \\ q_2^* \\ \vdots \end{pmatrix}(x) = \begin{pmatrix} 0 & \lambda\eta_1 & 0 & \cdots \\ 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} q_0^* \\ q_1^*(0) \\ q_2^*(0) \\ \vdots \end{pmatrix}$$
$$+ \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & \lambda\eta_1 & 0 & 0 & \cdots \\ 0 & 0 & 0 & \lambda\eta_1 & 0 & \cdots \\ 0 & 0 & 0 & 0 & \lambda\eta_1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} q_0^* \\ q_1^*(x) \\ q_2^*(x) \\ \vdots \end{pmatrix},$$

$$\mathbb{V}\begin{pmatrix} q_0^* \\ q_1^* \\ q_2^* \\ q_3^* \\ q_4^* \\ q_5^* \\ q_6^* \\ \vdots \end{pmatrix}(x) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \cdots \\ \mu(x)\eta_2 & \mu(x)q_2 & 0 & 0 & 0 & \cdots \\ 0 & \mu(x)\eta_2 & \mu(x)q_2 & 0 & 0 & \cdots \\ 0 & 0 & \mu(x)\eta_2 & \mu(x)q_2 & 0 & \cdots \\ 0 & 0 & 0 & \mu(x)\eta_2 & \mu(x)q_2 & \cdots \\ 0 & 0 & 0 & 0 & \mu(x)\eta_2 & \cdots \\ 0 & 0 & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} q_0^* \\ q_1^*(0) \\ q_2^*(0) \\ q_3^*(0) \\ q_4^*(0) \\ q_5^*(0) \\ q_6^*(0) \\ \vdots \end{pmatrix},$$

$$D(F) = D(\mathbb{V}) = X^*,$$

where α in $D(G)$ is a nonzero constant which is irrelevant to n.

Proof. For any $p \in D(A)$ and $q^* \in D(G)$ we have, integrating by parts and noting $p_n(\infty) = 0$ for $n \geq 1$,

$$\begin{aligned} &\langle (A+U+E)p, q^* \rangle \\ &= \left\{ -\lambda\eta_1 p_0 + \eta_2 \int_0^\infty p_1(x)\mu(x)dx \right\} q_0^* \end{aligned}$$

$$
\begin{aligned}
&+ \int_0^\infty \left\{ -\frac{dp_1(x)}{dx} - (\lambda\eta_1 + \mu(x))p_1(x) \right\} q_1^*(x)dx \\
&+ \sum_{n=2}^\infty \int_0^\infty \left\{ -\frac{dp_n(x)}{dx} - (\lambda\eta_1 + \mu(x))p_n(x) \right. \\
&\quad \left. + \lambda\eta_1 p_{n-1}(x) \right\} q_n^*(x)dx \\
= &-\lambda\eta_1 p_0 q_0^* + \eta_2 q_0^* \int_0^\infty p_1(x)\mu(x)dx \\
&+ \sum_{n=1}^\infty \int_0^\infty -\frac{dp_n(x)}{dx} q_n^*(x)dx \\
&- \sum_{n=1}^\infty \int_0^\infty (\lambda\eta_1 + \mu(x))p_n(x)q_n^*(x)dx \\
&+ \sum_{n=2}^\infty \int_0^\infty \lambda\eta_1 p_{n-1}(x)q_n^*(x)dx \\
= &-\lambda\eta_1 p_0 q_0^* + \eta_2 q_0^* \int_0^\infty p_1(x)\mu(x)dx \\
&- \sum_{n=1}^\infty p_n(x)q_n^*(x)\Big|_0^\infty + \sum_{n=1}^\infty \int_0^\infty p_n(x)\frac{dq_n^*(x)}{dx}dx \\
&- \sum_{n=1}^\infty \int_0^\infty (\lambda\eta_1 + \mu(x))p_n(x)q_n^*(x)dx \\
&+ \sum_{n=2}^\infty \int_0^\infty \lambda\eta_1 p_{n-1}(x)q_n^*(x)dx \\
= &-\lambda\eta_1 p_0 q_0^* + \eta_2 q_0^* \int_0^\infty p_1(x)\mu(x)dx \\
&+ \sum_{n=1}^\infty p_n(0)q_n^*(0) \\
&+ \sum_{n=1}^\infty \int_0^\infty \left[p_n(x)\frac{dq_n^*(x)}{dx} - (\lambda\eta_1 + \mu(x))p_n(x)q_n^*(x) \right] dx \\
&+ \sum_{n=2}^\infty \int_0^\infty \lambda\eta_1 p_{n-1}(x)q_n^*(x)dx \\
= &-\lambda\eta_1 p_0 q_0^* + \eta_2 q_0^* \int_0^\infty p_1(x)\mu(x)dx \\
&+ q_1^*(0) \left[\lambda\eta_1 p_0 + q_2 \int_0^\infty p_1(x)\mu(x)dx \right.
\end{aligned}
$$

$$
\begin{aligned}
&\quad +\eta_2 \int_0^\infty p_2(x)\mu(x)dx\Bigg] \\
&\quad + \sum_{n=2}^\infty q_n^*(0) \left[q_2 \int_0^\infty p_n(x)\mu(x)dx\right. \\
&\quad \left. +\eta_2 \int_0^\infty p_{n+1}(x)\mu(x)dx\right] \\
&\quad + \sum_{n=1}^\infty \int_0^\infty p_n(x) \left[\frac{dq_n^*(x)}{dx} - (\lambda\eta_1 + \mu(x))q_n^*(x)\right] dx \\
&\quad + \sum_{n=2}^\infty \int_0^\infty \lambda\eta_1 p_{n-1}(x) q_n^*(x)dx \\
&= -\lambda\eta_1 p_0 q_0^* + \lambda\eta_1 p_0 q_1^*(0) + \eta_2 q_0^* \int_0^\infty p_1(x)\mu(x)dx \\
&\quad + \sum_{n=1}^\infty q_n^*(0) \left[q_2 \int_0^\infty p_n(x)\mu(x)dx\right. \\
&\quad \left. +\eta_2 \int_0^\infty p_{n+1}(x)\mu(x)dx\right] \\
&\quad + \sum_{n=1}^\infty \int_0^\infty p_n(x) \left[\frac{dq_n^*(x)}{dx} - (\lambda\eta_1 + \mu(x))q_n^*(x)\right] dx \\
&\quad + \sum_{n=1}^\infty \int_0^\infty \lambda\eta_1 p_n(x) q_{n+1}^*(x)dx \\
&= -\lambda\eta_1 p_0 q_0^* \\
&\quad + \sum_{n=1}^\infty \int_0^\infty p_n(x) \left[\frac{dq_n^*(x)}{dx} - (\lambda\eta_1 + \mu(x))q_n^*(x)\right] \\
&\quad + \lambda\eta_1 p_0 q_1^*(0) + \sum_{n=1}^\infty \int_0^\infty \lambda\eta_1 p_n(x) q_{n+1}^*(x)dx \\
&\quad + \int_0^\infty p_1(x)\mu(x)\eta_2 q_0^* dx \\
&\quad + \sum_{n=1}^\infty \int_0^\infty p_{n+1}(x)\mu(x)\eta_2 q_n^*(0)dx \\
&\quad + \sum_{n=1}^\infty \int_0^\infty p_n(x)\mu(x) q_2 q_n^*(0)dx \\
&= \langle p, (G + F + \mathbb{V})q^* \rangle.
\end{aligned} \tag{4-47}
$$

From this, together with Definition 1.21, we know that the result of this lemma is right. □

In the following we consider the case $\mu(x) = \mu = \text{constant}$.

Lemma 4.5.
$$\left\{ \gamma \in \mathbb{C} \;\middle|\; \begin{array}{l} \Re\gamma + \lambda\eta_1 > 0,\ \max\left\{ \frac{\lambda\eta_1}{|\gamma+\lambda\eta_1|}, \right. \\ \left. \frac{\lambda\eta_1(|\gamma+\lambda\eta_1+\mu\eta_2|-\mu q_2)}{(\Re\gamma+\lambda\eta_1+\mu)(|\gamma+\lambda\eta_1+\mu\eta_2|-\mu\eta_2)} \right\} < 1 \end{array} \right\}$$
belongs to the resolvent set of $(A+U+E)^*$. *Particularly, all points on the imaginary axis except zero belong to the resolvent set of* $(A+U+E)^*$.

Proof. First consider the resolvent set of $G+\mathbb{V}$, then by using perturbation theory we obtain the desired result.

For any given $y^* \in X^*$ consider the equation $(G+\mathbb{V})q^* = y^*$, i.e.,

$$(\gamma + \lambda\eta_1)q_0^* = y_0^*, \tag{4-48}$$
$$\frac{dq_1^*(x)}{dx} = (\gamma + \lambda\eta_1 + \mu)q_1^*(x) - \mu\eta_2 q_0^* - \mu q_2 q_1^*(0) - y_1^*(x), \tag{4-49}$$
$$\frac{dq_n^*(x)}{dx} = (\gamma + \lambda\eta_1 + \mu)q_n^*(x) - \mu\eta_2 q_{n-1}^*(0) - \mu q_2 q_n^*(0) - y_n^*(x), \quad n \geq 2, \tag{4-50}$$
$$q_n^*(\infty) = \alpha, \quad n \geq 1. \tag{4-51}$$

Through solving (4-48)–(4-50) we have

$$q_0^* = \frac{1}{\gamma + \lambda\eta_1} y_0^*, \tag{4-52}$$
$$q_1^*(x) = a_1 e^{(\gamma+\lambda\eta_1+\mu)x} + e^{(\gamma+\lambda\eta_1+\mu)x} \int_0^x [-\mu\eta_2 q_0^* - \mu q_2 q_1^*(0) - y_1^*(\tau)] e^{-(\gamma+\lambda\eta_1+\mu)\tau} d\tau, \tag{4-53}$$
$$q_n^*(x) = a_n e^{(\gamma+\lambda\eta_1+\mu)x} + e^{(\gamma+\lambda\eta_1+\mu)x} \int_0^x [-\mu\eta_2 q_{n-1}^*(0) - \mu q_2 q_n^*(0) - y_n^*(\tau)] e^{-(\gamma+\lambda\eta_1+\mu)\tau} d\tau,\ n \geq 2. \tag{4-54}$$

By multiplying $e^{-(\gamma+\lambda\eta_1+\mu)x}$ to two sides of (4-53) and (4-54) and using (4-51) we obtain

$$a_1 = \int_0^\infty [\mu\eta_2 q_0^* + \mu q_2 q_1^*(0) + y_1^*(\tau)] e^{-(\gamma+\lambda\eta_1+\mu)\tau} d\tau, \tag{4-55}$$
$$a_n = \int_0^\infty [\mu\eta_2 q_{n-1}^*(0) + \mu q_2 q_n^*(0) + y_n^*(\tau)] \times e^{-(\gamma+\lambda\eta_1+\mu)\tau} d\tau, \quad n \geq 2. \tag{4-56}$$

Through inserting (4-55) and (4-56) into (4-53) and (4-54) respectively, it follows that (assume $\Re\gamma + \lambda\eta_1 + \mu > 0$)

$$\begin{aligned} q_1^*(x) &= e^{(\gamma+\lambda\eta_1+\mu)x} \int_x^\infty [\mu\eta_2 q_0^* + \mu q_2 q_1^*(0) \\ &\qquad + y_1^*(\tau)] e^{-(\gamma+\lambda\eta_1+\mu)\tau} d\tau \\ &= (\mu\eta_2 q_0^* + \mu q_2 q_1^*(0)) e^{(\gamma+\lambda\eta_1+\mu)x} \int_x^\infty e^{-(\gamma+\lambda\eta_1+\mu)\tau} d\tau \\ &\quad + e^{(\gamma+\lambda\eta_1+\mu)x} \int_x^\infty y_1^*(\tau) e^{-(\gamma+\lambda\eta_1+\mu)\tau} d\tau \\ &= \frac{1}{\gamma+\lambda\eta_1+\mu} (\mu\eta_2 q_0^* + \mu q_2 q_1^*(0)) \\ &\quad + e^{(\gamma+\lambda\eta_1+\mu)x} \int_x^\infty y_1^*(\tau) e^{-(\gamma+\lambda\eta_1+\mu)\tau} d\tau. \end{aligned} \tag{4-57}$$

$$\begin{aligned} q_n^*(x) &= e^{(\gamma+\lambda\eta_1+\mu)x} \int_x^\infty [\mu\eta_2 q_{n-1}^*(0) + \mu q_2 q_n^*(0) \\ &\qquad + y_n^*(\tau)] e^{-(\gamma+\lambda\eta_1+\mu)\tau} d\tau \\ &= \frac{1}{\gamma+\lambda\eta_1+\mu} (\mu\eta_2 q_{n-1}^*(0) + \mu q_2 q_n^*(0)) \\ &\quad + e^{(\gamma+\lambda\eta_1+\mu)x} \int_x^\infty y_n^*(\tau) e^{-(\gamma+\lambda\eta_1+\mu)\tau} d\tau, \ n \geq 2. \end{aligned} \tag{4-58}$$

By combining (4-57) with (4-52) and using $q_2 + \eta_2 = 1$ we derive

$$\begin{aligned} q_1^*(0) &= \frac{1}{\gamma+\lambda\eta_1+\mu} (\mu\eta_2 q_0^* + \mu q_2 q_1^*(0)) + \int_0^\infty y_1^*(\tau) e^{-(\gamma+\lambda\eta_1+\mu)\tau} d\tau \\ &\Longrightarrow \\ &\left[1 - \frac{\mu q_2}{\gamma+\lambda\eta_2+\mu}\right] q_1^*(0) \\ &= \frac{\mu\eta_2}{\gamma+\lambda\eta_1+\mu} q_0^* + \int_0^\infty y_1^*(\tau) e^{-(\gamma+\lambda\eta_1+\mu)\tau} d\tau \\ &\Longrightarrow \\ &\frac{\gamma+\lambda\eta_1+\mu-\mu q_2}{\gamma+\lambda\eta_1+\mu} q_1^*(0) \\ &= \frac{\mu\eta_2}{(\gamma+\lambda\eta_1)(\gamma+\lambda\eta_1+\mu)} y_0^* + \int_0^\infty y_1^*(\tau) e^{-(\gamma+\lambda\eta_1+\mu)\tau} d\tau \\ &\Longrightarrow \\ q_1^*(0) &= \frac{\mu\eta_2}{(\gamma+\lambda\eta_1)(\gamma+\lambda\eta_1+\mu\eta_2)} y_0^* \\ &\quad + \frac{\gamma+\lambda\eta_1+\mu}{\gamma+\lambda\eta_1+\mu\eta_2} \int_0^\infty y_1^*(\tau) e^{-(\gamma+\lambda\eta_1+\mu)\tau} d\tau. \end{aligned} \tag{4-59}$$

From (4-58), (4-59) and $\eta_2 + q_2 = 1$ we calculate

$$
\begin{aligned}
q_2^*(0) &= \frac{1}{\gamma+\lambda\eta_1+\mu}[\mu\eta_2 q_1^*(0) + \mu q_2 q_2^*(0)] \\
&\quad + \int_0^\infty y_2^*(\tau)e^{-(\gamma+\lambda\eta_1+\mu)\tau}d\tau \\
&\Longrightarrow \\
&\left[1 - \frac{\mu q_2}{\gamma+\lambda\eta_1+\mu}\right] q_2^*(0) \\
&= \frac{\mu\eta_2}{\gamma+\lambda\eta_1+\mu} q_1^*(0) + \int_0^\infty y_2^*(\tau)e^{-(\gamma+\lambda\eta_1+\mu)\tau}d\tau \\
&\Longrightarrow \\
&\frac{\gamma+\lambda\eta_1+\mu\eta_2}{\gamma+\lambda\eta_1+\mu} q_2^*(0) \\
&= \frac{\mu\eta_2}{\gamma+\lambda\eta_1+\mu}\left[\frac{\mu\eta_2}{(\gamma+\lambda\eta_1)(\gamma+\lambda\eta_1+\mu\eta_2)} y_0^*\right. \\
&\quad \left. + \frac{\gamma+\lambda\eta_1+\mu}{\gamma+\lambda\eta_1+\mu\eta_2}\int_0^\infty y_1^*(\tau)e^{-(\gamma+\lambda\eta_1+\mu)\tau}d\tau\right] \\
&\quad + \int_0^\infty y_2^*(\tau)e^{-(\gamma+\lambda\eta_1+\mu)\tau}d\tau \\
&\Longrightarrow \\
&\frac{\gamma+\lambda\eta_1+\mu\eta_2}{\gamma+\lambda\eta_1+\mu} q_2^*(0) \\
&= \frac{(\mu\eta_2)^2}{(\gamma+\lambda\eta_1)(\gamma+\lambda\eta_1+\mu)(\gamma+\lambda\eta_1+\mu\eta_2)} y_0^* \\
&\quad + \frac{\mu\eta_2}{\gamma+\lambda\eta_1+\mu\eta_2}\int_0^\infty y_1^*(\tau)e^{-(\gamma+\lambda\eta_1+\mu)\tau}d\tau \\
&\quad + \int_0^\infty y_2^*(\tau)e^{-(\gamma+\lambda\eta_1+\mu)\tau}d\tau \\
&\Longrightarrow \\
q_2^*(0) &= \frac{(\mu\eta_2)^2}{(\gamma+\lambda\eta_1)(\gamma+\lambda\eta_1+\mu\eta_2)^2} y_0^* \\
&\quad + \frac{\mu\eta_2(\gamma+\lambda\eta_1+\mu)}{(\gamma+\lambda\eta_1+\mu\eta_2)^2}\int_0^\infty y_1^*(\tau)e^{-(\gamma+\lambda\eta_1+\mu)\tau}d\tau \\
&\quad + \frac{\gamma+\lambda\eta_1+\mu}{\gamma+\lambda\eta_1+\mu\eta_2}\int_0^\infty y_2^*(\tau)e^{-(\gamma+\lambda\eta_1+\mu)\tau}d\tau. \qquad (4\text{-}60)
\end{aligned}
$$

By inserting (4-60) into (4-58) and using $q_2 + \eta_2 = 1$ we deduce

$$
\begin{aligned}
q_3^*(0) &= \frac{\mu\eta_2 q_2^*(0) + \mu q_2 q_3^*(0)}{\gamma + \lambda\eta_1 + \mu} + \int_0^\infty y_3^*(\tau) e^{-(\gamma+\lambda\eta_1+\mu)\tau} d\tau \\
&\Longrightarrow \\
&\frac{\gamma + \lambda\eta_1 + \mu\eta_2}{\gamma + \lambda\eta_1 + \mu} q_3^*(0) \\
&= \frac{\mu\eta_2}{\gamma + \lambda\eta_1 + \mu} q_2^*(0) + \int_0^\infty y_3^*(\tau) e^{-(\gamma+\lambda\eta_1+\mu)\tau} d\tau \\
&\Longrightarrow \\
q_3^*(0) &= \frac{\mu\eta_2}{\gamma + \lambda\eta_1 + \mu\eta_2} q_2^*(0) \\
&\quad + \frac{\gamma + \lambda\eta_1 + \mu}{\gamma + \lambda\eta_1 + \mu\eta_2} \int_0^\infty y_3^*(\tau) e^{-(\gamma+\lambda\eta_1+\mu)\tau} d\tau \\
&= \frac{(\mu\eta_2)^3}{(\gamma + \lambda\eta_1)(\gamma + \lambda\eta_1 + \mu\eta_2)^3} y_0^* \\
&\quad + \frac{(\mu\eta_2)^2(\gamma + \lambda\eta_1 + \mu)}{(\gamma + \lambda\eta_1 + \mu\eta_2)^3} \int_0^\infty y_1^*(\tau) e^{-(\gamma+\lambda\eta_1+\mu)\tau} d\tau \\
&\quad + \frac{\mu\eta_2(\gamma + \lambda\eta_1 + \mu)}{(\gamma + \lambda\eta_1 + \mu\eta_2)^2} \int_0^\infty y_2^*(\tau) e^{-(\gamma+\lambda\eta_1+\mu)\tau} d\tau \\
&\quad + \frac{\gamma + \lambda\eta_1 + \mu}{\gamma + \lambda\eta_1 + \mu\eta_2} \int_0^\infty y_3^*(\tau) e^{-(\gamma+\lambda\eta_1+\mu)\tau} d\tau. \qquad (4\text{-}61)
\end{aligned}
$$

It is the same as (4-59), (4-60) and (4-61); by applying (4-61) in (4-58) it follows, by induction,

$$
\begin{aligned}
q_n^*(0) = &\frac{(\mu\eta_2)^n}{(\gamma + \lambda\eta_1)(\gamma + \lambda\eta_1 + \mu\eta_2)^n} y_0^* \\
&+ \frac{(\mu\eta_2)^{n-1}(\gamma + \lambda\eta_1 + \mu)}{(\gamma + \lambda\eta_1 + \mu\eta_2)^n} \int_0^\infty y_1^*(\tau) e^{-(\gamma+\lambda\eta_1+\mu)\tau} d\tau \\
&+ \frac{(\mu\eta_2)^{n-2}(\gamma + \lambda\eta_1 + \mu)}{(\gamma + \lambda\eta_1 + \mu\eta_2)^{n-1}} \int_0^\infty y_2^*(\tau) e^{-(\gamma+\lambda\eta_1+\mu)\tau} d\tau \\
&+ \frac{(\mu\eta_2)^{n-3}(\gamma + \lambda\eta_1 + \mu)}{(\gamma + \lambda\eta_1 + \mu\eta_2)^{n-2}} \int_0^\infty y_3^*(\tau) e^{-(\gamma+\lambda\eta_1+\mu)\tau} d\tau \\
&+ \cdots\cdots \\
&+ \frac{(\mu\eta_2)^2(\gamma + \lambda\eta_1 + \mu)}{(\gamma + \lambda\eta_1 + \mu\eta_2)^3} \int_0^\infty y_{n-2}^*(\tau) e^{-(\gamma+\lambda\eta_1+\mu)\tau} d\tau \\
&+ \frac{\mu\eta_2(\gamma + \lambda\eta_1 + \mu)}{(\gamma + \lambda\eta_1 + \mu\eta_2)^2} \int_0^\infty y_{n-1}^*(\tau) e^{-(\gamma+\lambda\eta_1+\mu)\tau} d\tau
\end{aligned}
$$

$$+\frac{\gamma+\lambda\eta_1+\mu}{\gamma+\lambda\eta_1+\mu\eta_2}\int_0^\infty y_n^*(\tau)e^{-(\gamma+\lambda\eta_1+\mu)\tau}d\tau, \quad n\geq 1. \tag{4-62}$$

By (4-62) it is easy to estimate; without loss of generality assume $\Re\gamma+\lambda\eta_1>0$,

$$\begin{aligned}
|q_n^*(0)| \leq & \frac{(\mu\eta_2)^n}{|\gamma+\lambda\eta_1||\gamma+\lambda\eta_1+\mu\eta_2|^n}|y_0^*| \\
& + \frac{(\mu\eta_2)^{n-1}}{|\gamma+\lambda\eta_1+\mu\eta_2|^n}\frac{|\gamma+\lambda\eta_1+\mu|}{\Re\gamma+\lambda\eta_1+\mu}\|y_1^*\|_{L^\infty[0,\infty)} \\
& + \frac{(\mu\eta_2)^{n-2}}{|\gamma+\lambda\eta_1+\mu\eta_2|^{n-1}}\frac{|\gamma+\lambda\eta_1+\mu|}{\Re\gamma+\lambda\eta_1+\mu}\|y_2^*\|_{L^\infty[0,\infty)} \\
& + \frac{(\mu\eta_2)^{n-3}}{|\gamma+\lambda\eta_1+\mu\eta_2|^{n-2}}\frac{|\gamma+\lambda\eta_1+\mu|}{\Re\gamma+\lambda\eta_1+\mu}\|y_3^*\|_{L^\infty[0,\infty)} \\
& + \cdots\cdots \\
& + \frac{(\mu\eta_2)^2}{|\gamma+\lambda\eta_1+\mu\eta_2|^3}\frac{|\gamma+\lambda\eta_1+\mu|}{\Re\gamma+\lambda\eta_1+\mu}\|y_{n-2}^*\|_{L^\infty[0,\infty)} \\
& + \frac{\mu\eta_2}{|\gamma+\lambda\eta_1+\mu\eta_2|^2}\frac{|\gamma+\lambda\eta_1+\mu|}{\Re\gamma+\lambda\eta_1+\mu}\|y_{n-1}^*\|_{L^\infty[0,\infty)} \\
& + \frac{1}{|\gamma+\lambda\eta_1+\mu\eta_2|}\frac{|\gamma+\lambda\eta_1+\mu|}{\Re\gamma+\lambda\eta_1+\mu}\|y_n^*\|_{L^\infty[0,\infty)} \\
\leq & \left\{\frac{(\mu\eta_2)^n}{|\gamma+\lambda\eta_1||\gamma+\lambda\eta_1+\mu\eta_2|^n}\right. \\
& + \frac{(\mu\eta_2)^{n-1}}{|\gamma+\lambda\eta_1+\mu\eta_2|^n}\frac{|\gamma+\lambda\eta_1+\mu|}{\Re\gamma+\lambda\eta_1+\mu} \\
& + \frac{(\mu\eta_2)^{n-2}}{|\gamma+\lambda\eta_1+\mu\eta_2|^{n-1}}\frac{|\gamma+\lambda\eta_1+\mu|}{\Re\gamma+\lambda\eta_1+\mu} \\
& + \frac{(\mu\eta)^{n-3}}{|\gamma+\lambda\eta_1+\mu\eta_2|^{n-2}}\frac{|\gamma+\lambda\eta_1+\mu|}{\Re\gamma+\lambda\eta_1+\mu} \\
& + \cdots\cdots \\
& + \frac{(\mu\eta_2)^2}{|\gamma+\lambda\eta_1+\mu\eta_2|^3}\frac{|\gamma+\lambda\eta_1+\mu|}{\Re\gamma+\lambda\eta_1+\mu} \\
& + \frac{\mu\eta_2}{|\gamma+\lambda\eta_1+\mu\eta_2|^2}\frac{|\gamma+\lambda\eta_1+\mu|}{\Re\gamma+\lambda\eta_1+\mu} \\
& \left. + \frac{1}{|\gamma+\lambda\eta_1+\mu\eta_2|}\frac{|\gamma+\lambda\eta_1+\mu|}{\Re\gamma+\lambda\eta_1+\mu}\right\} |||y^*||| \\
= & \left\{\frac{(\mu\eta_2)^n}{|\gamma+\lambda\eta_1||\gamma+\lambda\eta_1+\mu\eta_2|^n}\right. \\
& + \frac{|\gamma+\lambda\eta_1+\mu|}{\Re\gamma+\lambda\eta_1+\mu}\frac{\frac{1}{|\gamma+\lambda\eta_1+\mu\eta_2|}}{1-\frac{\mu\eta_2}{|\gamma+\lambda\eta_1+\mu\eta_2|}}
\end{aligned}$$

$$
\begin{aligned}
&\times \left[1-\left(\frac{\mu\eta_2}{|\gamma+\lambda\eta_1+\mu\eta_2|}\right)^n\right]\Bigg\} |||y^*||| \\
&= \Bigg\{\frac{(\mu\eta_2)^n}{|\gamma+\lambda\eta_1||\gamma+\lambda\eta_1+\mu\eta_2|^n} \\
&\quad + \frac{|\gamma+\lambda\eta_1+\mu|}{\Re\gamma+\lambda\eta_1+\mu}\frac{1}{|\gamma+\lambda\eta_1+\mu\eta_2|-\mu\eta_2} \\
&\quad \times \left[1-\left(\frac{\mu\eta_2}{|\gamma+\lambda\eta_1+\mu\eta_2|}\right)^n\right]\Bigg\} |||y^*||| \\
&= \Bigg\{\frac{(\mu\eta_2)^n}{|\gamma+\lambda\eta_1||\gamma+\lambda\eta_1+\mu\eta_2|^n} \\
&\quad + \frac{|\gamma+\lambda\eta_1+\mu|}{\Re\gamma+\lambda\eta_1+\mu}\frac{1}{|\gamma+\lambda\eta_1+\mu\eta_2|-\mu\eta_2} \\
&\quad - \frac{|\gamma+\lambda\eta_1+\mu|}{\Re\gamma+\lambda\eta_1+\mu}\frac{1}{|\gamma+\lambda\eta_1+\mu\eta_2|-\mu\eta_2} \\
&\quad \times \left(\frac{\mu\eta_2}{|\gamma+\lambda\eta_1+\mu\eta_2|}\right)^n\Bigg\} |||y^*|||, \quad n \geq 1.
\end{aligned} \tag{4-63}
$$

By using (4-63) and the relation $q_2+\eta_2=1$ we get

$$
\begin{aligned}
&\mu\eta_2|q_{n-1}^*(0)|+\mu q_2|q_n^*(0)| \\
&\leq \Bigg\{\frac{(\mu\eta_2)^n}{|\gamma+\lambda\eta_1||\gamma+\lambda\eta_1+\mu\eta_2|^{n-1}} \\
&\quad + \frac{\mu\eta_2}{|\gamma+\lambda\eta_1+\mu\eta_2|-\mu\eta_2}\frac{|\gamma+\lambda\eta_1+\mu|}{\Re\gamma+\lambda\eta_1+\mu} \\
&\quad - \frac{|\gamma+\lambda\eta_1+\mu|}{\Re\gamma+\lambda\eta_1+\mu}\frac{\mu\eta_2}{|\gamma+\lambda\eta_1+\mu\eta_2|-\mu\eta_2} \\
&\quad \times \left(\frac{\mu\eta_2}{|\gamma+\lambda\eta_1+\mu\eta_2|}\right)^{n-1}\Bigg\} |||y^*||| \\
&\quad + \Bigg\{\frac{\mu q_2(\mu\eta_2)^n}{|\gamma+\lambda\eta_1||\gamma+\lambda\eta_1+\mu\eta_2|^n} \\
&\quad + \frac{|\gamma+\lambda\eta_1+\mu|}{\Re\gamma+\lambda\eta_1+\mu}\frac{\mu q_2}{|\gamma+\lambda\eta_1+\mu\eta_2|-\mu\eta_2} \\
&\quad - \frac{|\gamma+\lambda\eta_1+\mu|}{\Re\gamma+\lambda\eta_1+\mu}\frac{\mu q_2}{|\gamma+\lambda\eta_1+\mu\eta_2|-\mu\eta_2} \\
&\quad \times \left(\frac{\mu\eta_2}{|\gamma+\lambda\eta_1+\mu\eta_2|}\right)^n\Bigg\} |||y^*||| \\
&= \Bigg\{\frac{(\mu\eta_2)^n}{|\gamma+\lambda\eta_1||\gamma+\lambda\eta_1+\mu\eta_2|^{n-1}}
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{\mu q_2(\mu\eta_2)^n}{|\gamma+\lambda\eta_1||\gamma+\lambda\eta_1+\mu\eta_2|^n}\\
&+\frac{|\gamma+\lambda\eta_1+\mu|}{\Re\gamma+\lambda\eta_1+\mu}\frac{\mu}{|\gamma+\lambda\eta_1+\mu\eta_2|-\mu\eta_2}\\
&-\frac{|\gamma+\lambda\eta_1+\mu|}{\Re\gamma+\lambda\eta_1+\mu}\frac{\mu}{|\gamma+\lambda\eta_1+\mu\eta_2|-\mu\eta_2}\\
&\quad\times\left(\frac{\mu\eta_2}{|\gamma+\lambda\eta_1+\mu\eta_2|}\right)^{n-1}\\
&\quad\times\left[\eta_2+q_2\frac{\mu\eta_2}{|\gamma+\lambda\eta_1+\mu\eta_2|}\right]\Bigg\}|||y^*|||\\
=&\left\{\frac{(\mu\eta_2)^n}{|\gamma+\lambda\eta_1||\gamma+\lambda\eta_1+\mu\eta_2|^{n-1}}\left[1+\frac{\mu q_2}{|\gamma+\lambda\eta_1+\mu\eta_2|}\right]\right.\\
&+\frac{|\gamma+\lambda\eta_1+\mu|}{\Re\gamma+\lambda\eta_1+\mu}\frac{\mu}{|\gamma+\lambda\eta_1+\mu\eta_2|-\mu\eta_2}\\
&-\frac{|\gamma+\lambda\eta_1+\mu|}{\Re\gamma+\lambda\eta_1+\mu}\frac{\mu\eta_2}{|\gamma+\lambda\eta_1+\mu\eta_2|-\mu\eta_2}\\
&\quad\times\left(\frac{\mu\eta_2}{|\gamma+\lambda\eta_1+\mu\eta_2|}\right)^{n-1}\\
&\quad\times\left.\left[1+\frac{\mu q_2}{|\gamma+\lambda\eta_1+\mu\eta_2|}\right]\right\}|||y^*|||,\quad n\geq 2.
\end{aligned}\tag{4-64}
$$

By combining (4-58) with (4-64) and using $q_2+\eta_2=1$ we estimate

$$
\begin{aligned}
&\|q_n^*\|_{L^\infty[0,\infty)}\\
&\leq\frac{1}{|\gamma+\lambda\eta_1+\mu|}\{\mu\eta_2|q_{n-1}^*(0)|+\mu q_2|q_n^*(0)|\}\\
&\quad+\frac{1}{\Re\gamma+\lambda\eta_1+\mu}\|y_n^*\|_{L^\infty[0,\infty)}\\
&\leq\left\{\frac{1}{|\gamma+\lambda\eta_1||\gamma+\lambda\eta_1+\mu|}\frac{(\mu\eta_2)^n}{|\gamma+\lambda\eta_1+\mu\eta_2|^{n-1}}\right.\\
&\qquad\times\left[1+\frac{\mu q_2}{|\gamma+\lambda\eta_1+\mu\eta_2|}\right]\\
&\quad+\frac{\mu}{(\Re\gamma+\lambda\eta_1+\mu)(|\gamma+\lambda\eta_1+\mu\eta_2|-\mu\eta_2)}\\
&\quad-\frac{\mu\eta_2}{(\Re\gamma+\lambda\eta_1+\mu)(|\gamma+\lambda\eta_1+\mu\eta_2|-\mu\eta_2)}\\
&\qquad\times\left.\left(\frac{\mu\eta_2}{|\gamma+\lambda\eta_1+\mu\eta_2|}\right)^{n-1}\left[1+\frac{\mu q_2}{|\gamma+\lambda\eta_1+\mu\eta_2|}\right]\right\}|||y^*|||\\
&\quad+\frac{1}{\Re\gamma+\lambda\eta_1+\mu}\|y_n^*\|_{L^\infty[0,\infty)}
\end{aligned}
$$

$$
\begin{aligned}
&\leq \left\{ \frac{1}{|\gamma+\lambda\eta_1||\gamma+\lambda\eta_1+\mu|} \frac{(\mu\eta_2)^n}{|\gamma+\lambda\eta_1+\mu\eta_2|^{n-1}} \right.\\
&\qquad \times \left[1+\frac{\mu q_2}{|\gamma+\lambda\eta_1+\mu\eta_2|}\right]\\
&\quad + \frac{\mu}{(\Re\gamma+\lambda\eta_1+\mu)(|\gamma+\lambda\eta_1+\mu\eta_2|-\mu\eta_2)}\\
&\quad - \frac{\mu\eta_2}{(\Re\gamma+\lambda\eta_1+\mu)(|\gamma+\lambda\eta_1+\mu\eta_2|-\mu\eta_2)}\\
&\qquad \left. \times \left(\frac{\mu\eta_2}{|\gamma+\lambda\eta_1+\mu\eta_2|}\right)^{n-1} \left[1+\frac{\mu q_2}{|\gamma+\lambda\eta_1+\mu\eta_2|}\right] \right\} |||y^*|||\\
&\quad + \frac{1}{\Re\gamma+\lambda\eta_1+\mu}|||y^*|||\\
&= \left\{ \frac{1}{|\gamma+\lambda\eta_1||\gamma+\lambda\eta_1+\mu|} \frac{(\mu\eta_2)^n}{|\gamma+\lambda\eta_1+\mu\eta_2|^{n-1}} \right.\\
&\qquad \times \left[1+\frac{\mu q_2}{|\gamma+\lambda\eta_1+\mu\eta_2|}\right]\\
&\quad + \frac{|\gamma+\lambda\eta_1+\mu\eta_2|+\mu q_2}{(\Re\gamma+\lambda\eta_1+\mu)(|\gamma+\lambda\eta_1+\mu\eta_2|-\mu\eta_2)}\\
&\quad - \frac{\mu\eta_2}{(\Re\gamma+\lambda\eta_1+\mu)(|\gamma+\lambda\eta_1+\mu\eta_2|-\mu\eta_2)}\\
&\qquad \times \left(\frac{\mu\eta_2}{|\gamma+\lambda\eta_1+\mu\eta_2|}\right)^{n-1}\\
&\qquad \left. \left[1+\frac{\mu q_2}{|\gamma+\lambda\eta_1+\mu\eta_2|}\right] \right\} |||y^*|||, \quad n \geq 1. \qquad (4\text{-}65)
\end{aligned}
$$

It is obvious that the following inequalities hold.

$$\Re\gamma+\lambda\eta_1+\mu \leq |\gamma+\lambda\eta_1+\mu|. \qquad (4\text{-}66)$$

$$|\gamma+\lambda\eta_1+\mu\eta_2|-\mu\eta_2 \leq |\gamma+\lambda\eta_1|. \qquad (4\text{-}67)$$

From (4-66) and (4-67) it is easy to derive

$$
\begin{aligned}
&(\Re\gamma+\lambda\eta_1+\mu)(|\gamma+\lambda\eta_1+\mu\eta_2|-\mu\eta_2)\\
&\leq |\gamma+\lambda\eta_1||\gamma+\lambda\eta_1+\mu|.
\end{aligned}
$$

This implies

$$
\begin{aligned}
&\frac{1}{|\gamma+\lambda\eta_1||\gamma+\lambda\eta_1+\mu|} \frac{(\mu\eta_2)^{n-1}}{|\gamma+\lambda\eta_1+\mu\eta_2|^{n-2}}\\
&\quad \times \left[1+\frac{\mu q_2}{|\gamma+\lambda\eta_1+\mu\eta_2|}\right]
\end{aligned}
$$

$$
\begin{aligned}
&\quad - \frac{\mu\eta_2}{(\Re\gamma+\lambda\eta_1+\mu)(|\gamma+\lambda\eta_1+\mu\eta_2|-\mu\eta_2)} \\
&\qquad \times \left(\frac{\mu\eta_2}{|\gamma+\lambda\eta_1+\mu\eta_2|}\right)^{n-2}\left[1+\frac{\mu q_2}{|\gamma+\lambda\eta_1+\mu\eta_2|}\right] \\
&\leq \frac{1}{|\gamma+\lambda\eta_1||\gamma+\lambda\eta_1+\mu|}\frac{(\mu\eta_2)^n}{|\gamma+\lambda\eta_1+\mu\eta_2|^{n-1}} \\
&\qquad \times \left[1+\frac{\mu q_2}{|\gamma+\lambda\eta_1+\mu\eta_2|}\right] \\
&\quad - \frac{\mu\eta_2}{(\Re\gamma+\lambda\eta_1+\mu)(|\gamma+\lambda\eta_1+\mu\eta_2|-\mu\eta_2)} \\
&\qquad \times \left(\frac{\mu\eta_2}{|\gamma+\lambda\eta_1+\mu\eta_2|}\right)^{n-1} \\
&\qquad \times \left[1+\frac{\mu q_2}{|\gamma+\lambda\eta_1+\mu\eta_2|}\right], \quad n \geq 2.
\end{aligned} \tag{4-68}
$$

By using (4-68) we simplify (4-65) as follows.

$$
\begin{aligned}
&\sup_{n\geq 1}\|q_n^*\|_{L^\infty[0,\infty)} \\
&\leq \lim_{n\to\infty}\left\{\frac{1}{|\gamma+\lambda\eta_1||\gamma+\lambda\eta_1+\mu|}\frac{(\mu\eta_2)^n}{|\gamma+\lambda\eta_1+\mu\eta_2|^{n-1}}\right. \\
&\qquad \times \left[1+\frac{\mu q_2}{|\gamma+\lambda\eta_1+\mu\eta_2|}\right] \\
&\quad + \frac{|\gamma+\lambda\eta_1+\mu\eta_2|+\mu q_2}{(\Re\gamma+\lambda\eta_1+\mu)(|\gamma+\lambda\eta_1+\mu\eta_2|-\mu\eta_2)} \\
&\quad - \frac{\mu\eta_2}{(\Re\gamma+\lambda\eta_1+\mu)(|\gamma+\lambda\eta_1+\mu\eta_2|-\mu\eta_2)} \\
&\qquad \left.\times \left(\frac{\mu\eta_2}{|\gamma+\lambda\eta_1+\mu\eta_2|}\right)^{n-1}\left[1+\frac{\mu q_2}{|\gamma+\lambda\eta_1+\mu\eta_2|}\right]\right\}|||y^*||| \\
&= \frac{|\gamma+\lambda\eta_1+\mu\eta_2|+\mu q_2}{(\Re\gamma+\lambda\eta_1+\mu)(|\gamma+\lambda\eta_1+\mu\eta_2|-\mu\eta_2)}|||y^*|||.
\end{aligned} \tag{4-69}
$$

By combining (4-69) with (4-52) we deduce

$$
\begin{aligned}
|||q^*||| &= \max\{|q_0^*|, \sup_{n\geq 1}\|q_n^*\|_{L^\infty[0,\infty)}\} \\
&\leq \max\left\{\frac{1}{|\gamma+\lambda\eta_1|},\right. \\
&\quad \left.\frac{|\gamma+\lambda\eta_1+\mu\eta_2|+\mu q_2}{(\Re\gamma+\lambda\eta_1+\mu)(|\gamma+\lambda\eta_1+\mu\eta_2|-\mu\eta_2)}\right\}|||y^*|||.
\end{aligned} \tag{4-70}
$$

(4-70) shows that $[\gamma I-(G+\mathbb{V})]^{-1}$ exists and

$$\begin{aligned}
&\|[\gamma I-(G+\mathbb{V})]^{-1}\| \\
&\le \max\left\{\frac{1}{|\gamma+\lambda\eta_1|},\right. \\
&\qquad \left.\frac{|\gamma+\lambda\eta_1+\mu\eta_2|+\mu q_2}{(\Re\gamma+\lambda\eta_1+\mu)(|\gamma+\lambda\eta_1+\mu\eta_2|-\mu\eta_2)}\right\}
\end{aligned} \tag{4-71}$$

when $\Re\gamma+\lambda\eta_1>0$.

In the following we will estimate the norm of F. From the definition of F we have, for any $q^*\in X^*$,

$$\begin{aligned}
Fq^*(x) &= \begin{pmatrix} \lambda\eta_1 q_1^*(0) \\ \lambda\eta_1 q_2^*(x) \\ \lambda\eta_1 q_3^*(x) \\ \vdots \end{pmatrix} \\
&\Longrightarrow \\
|||Fq^*||| &= \max\left\{|\lambda\eta_1 q_1^*(0)|, \sup_{n\ge1}\|\lambda\eta_1 q_n^*(x)\|_{L^\infty[0,\infty)}\right\} \\
&= \lambda\eta_1 \max\left\{|q_1^*(0)|, \sup_{n\ge1}\|q_n^*\|_{L^\infty[0,\infty)}\right\} \\
&\le \lambda\eta_1 \max\left\{\|q_1^*\|_{L^\infty[0,\infty)}, \sup_{n\ge1}\|q_n^*\|_{L^\infty[0,\infty)}\right\} \\
&\le \lambda\eta_1|||q^*|||.
\end{aligned} \tag{4-72}$$

On the other hand, we take a special $y^*(x)=(0,1,0,0,\ldots)\in X^*$, then from (4-72) we obtain

$$|||Fy^*||| = \max\{\lambda\eta_1, 0\} = \lambda\eta_1|||y^*|||. \tag{4-73}$$

(4-72) and (4-73) give

$$\|F\| = \lambda\eta_1. \tag{4-74}$$

Since (4-71) implies $[I-(\gamma I-G-\mathbb{V})^{-1}F]^{-1}$ exists and is bounded when

$$\begin{aligned}
\|(\gamma I-G-\mathbb{V})^{-1}F\| &\le \|(\gamma I-G-\mathbb{V})^{-1}\|\|F\| \\
&\le \max\left\{\frac{\lambda\eta_1}{|\gamma+\lambda\eta_1|},\right. \\
&\qquad \left.\frac{\lambda\eta_1[|\gamma+\lambda\eta_1+\mu\eta_2|+\mu q_2]}{(\Re\gamma+\lambda\eta_1+\mu)(|\gamma+\lambda\eta_1+\mu\eta_2|-\mu\eta_2)}\right\} \\
&< 1,
\end{aligned} \tag{4-75}$$

by using the formula

$$
\begin{aligned}
&[\gamma I-(A+U+E)^*]^{-1}\\
&=[\gamma I-(G+F+\mathbb{V})]^{-1}\\
&=[I-(\gamma I-G-\mathbb{V})^{-1}F]^{-1}(\gamma I-G-\mathbb{V})^{-1}
\end{aligned}
$$

we know that $[\gamma I-(A+U+E)^*]^{-1}$ exists and is bounded when (4-75) holds, i.e., the set

$$
\left\{\gamma\in\mathbb{C}\;\middle|\;\begin{array}{l}\Re\gamma+\lambda\eta_1>0,\quad \max\left\{\frac{\lambda\eta_1}{|\gamma+\lambda\eta_1|},\right.\\ \left.\frac{\lambda\eta_1[|\gamma+\lambda\eta_1+\mu\eta_2|+\mu q_2]}{(\Re\gamma+\lambda\eta_1+\mu)(|\gamma+\lambda\eta_1+\mu\eta_2|-\mu\eta_2)}\right\}\\ <1\end{array}\right\}
$$

belongs to the resolvent set of $(A+U+E)^*$ (see Definition 1.22).

Particularly, if $\gamma=ia,\ a\neq 0$, then (4-75) automatically holds. In fact, using $q_2+\eta_2=1$,

$$
\begin{aligned}
&a^2>0\Longrightarrow(\lambda\eta_1)^2<(\lambda\eta_1)^2+a^2\Longrightarrow\\
&\lambda\eta_1<\sqrt{(\lambda\eta)^2+a^2}\Longrightarrow\frac{\lambda\eta_1}{\sqrt{(\lambda\eta_1)^2+a^2}}<1.
\end{aligned}\tag{4-76}
$$

$$
\begin{aligned}
&a^2>0\Longrightarrow\lambda\eta_1+\mu\eta_2<\sqrt{a^2+(\lambda\eta_1+\mu\eta_2)^2}\\
&\Longrightarrow\\
&\lambda\eta_1\mu(q_2+\eta_2)+\mu^2\eta_2<\mu\sqrt{a^2+(\lambda\eta_1+\mu\eta_2)^2}\\
&\Longrightarrow\\
&\lambda\eta_1\mu q_2+\lambda\eta_1\mu\eta_2+\mu^2\eta_2<\mu\sqrt{a^2+(\lambda\eta_1+\mu\eta_2)^2}\\
&\Longrightarrow\\
&\lambda\eta_1\mu q_2+\lambda\eta_1\sqrt{a^2+(\lambda\eta_1+\mu\eta_2)^2}\\
&<\lambda\eta_1\sqrt{a^2+(\lambda\eta_1+\mu\eta_2)^2}-\lambda\eta_1\mu\eta_2\\
&\qquad+\mu\sqrt{a^2+(\lambda\eta_1+\mu\eta_2)^2}-\mu^2\eta_2\\
&\Longrightarrow\\
&\lambda\eta_1\left[\mu q_2+\sqrt{a^2+(\lambda\eta_1+\mu\eta_2)^2}\right]\\
&<(\lambda\eta_1+\mu)\left(\sqrt{a^2+(\lambda\eta_1+\mu\eta_2)^2}-\mu\eta_2\right)\\
&\Longrightarrow\\
&\frac{\lambda\eta_1\left[\mu q_2+\sqrt{a^2+(\lambda\eta_1+\mu\eta_2)^2}\right]}{(\lambda\eta_1+\mu)\left(\sqrt{a^2+(\lambda\eta_1+\mu\eta_2)^2}-\mu\eta_2\right)}<1.
\end{aligned}\tag{4-77}
$$

(4-76) and (4-77) imply

$$\max\left\{\frac{\lambda\eta_1}{\sqrt{a^2+(\lambda\eta_1)^2}}, \frac{\lambda\eta_1[\mu q_2+\sqrt{a^2+(\lambda\eta_1+\mu\eta_2)^2}]}{(\lambda\eta_1+\mu)(\sqrt{a^2+(\lambda\eta_1+\mu\eta_2)^2}-\mu\eta_2)}\right\}<1,$$

from which together with $\Re\gamma+\lambda\eta_1 = \lambda\eta_1 > 0$ for all $\gamma = ia$, $a \neq 0$ we know that all points on the imaginary axis except zero belong to the resolvent of $(A+U+E)^*$. □

Lemma 4.6. *If $\lambda\eta_1 < \mu\eta_2$, then* 0 *is an eigenvalue of $A+U+E$ with geometric multiplicity* 1*.*

Proof. Consider the equation $(A+U+E)p = 0$. It is equivalent to

$$-\lambda\eta_1 p_0+\mu\eta_2\int_0^\infty p_1(x)dx = 0, \tag{4-78}$$

$$\frac{dp_1(x)}{dx} = -(\lambda\eta_1+\mu)p_1(x), \tag{4-79}$$

$$\frac{dp_n(x)}{dx} = -(\lambda\eta_1+\mu)p_n(x)+\lambda\eta_1 p_{n-1}(x), \quad n\geq 2, \tag{4-80}$$

$$p_1(0) = \lambda\eta_1 p_0+\mu q_2\int_0^\infty p_1(x)dx+\mu\eta_2\int_0^\infty p_2(x)dx, \tag{4-81}$$

$$p_n(0) = \mu q_2\int_0^\infty p_n(x)dx+\mu\eta_2\int_0^\infty p_{n+1}(x)dx,\ n\geq 2. \tag{4-82}$$

By solving (4-79) and (4-80) we obtain

$$p_1(x) = b_1 e^{-(\lambda\eta_1+\mu)x}, \tag{4-83}$$

$$p_n(x) = b_n e^{-(\lambda\eta_1+\mu)x} + \lambda\eta_1 e^{-(\lambda\eta_1+\mu)x}\int_0^x p_{n-1}(\tau)e^{(\lambda\eta_1+\mu)\tau}d\tau,\ n\geq 2. \tag{4-84}$$

Through inserting (4-83) into (4-78) we deduce

$$\lambda\eta_1 p_0 = \mu\eta_2\int_0^\infty b_1 e^{-(\lambda\eta_1+\mu)x}dx = b_1\frac{\mu\eta_2}{\lambda\eta_1+\mu}$$

$$\Longrightarrow$$

$$b_1 = \frac{\lambda\eta_1+\mu}{\mu\eta_2}\lambda\eta_1 p_0. \tag{4-85}$$

By combining (4-85), (4-83) and (4-84) with (4-81) and using the Fubini theorem (see Theorem 1.49) and the relation $q_2+\eta_2 = 1$ we derive

$$b_1 = \lambda\eta_1 p_0+\frac{\mu q_2}{\lambda\eta_1+\mu}b_1+\frac{\mu\eta_2}{\lambda\eta_1+\mu}b_2$$

$$
\begin{aligned}
&+ \mu\eta_2\lambda\eta_1 \int_0^\infty e^{-(\lambda\eta_1+\mu)x} \int_0^x p_1(\tau)e^{(\lambda\eta_1+\mu)\tau} d\tau dx \\
&= \lambda\eta_1 p_0 + \frac{\mu q_2}{\lambda\eta_1+\mu} b_1 + \frac{\mu\eta_2}{\lambda\eta_1+\mu} b_2 \\
&\quad + \mu\eta_2\lambda\eta_1 \int_0^\infty p_1(\tau)e^{(\lambda\eta_1+\mu)\tau} \int_\tau^\infty e^{-(\lambda\eta_1+\mu)x} dx d\tau \\
&= \lambda\eta_1 p_0 + \frac{\mu q_2}{\lambda\eta_1+\mu} b_1 + \frac{\mu\eta_2}{\lambda\eta_1+\mu} b_2 \\
&\quad + \frac{\mu\eta_2\lambda\eta_1}{\lambda\eta_1+\mu} \int_0^\infty p_1(\tau) d\tau \\
&= \lambda\eta_1 p_0 + \frac{\mu q_2}{\lambda\eta_1+\mu} b_1 + \frac{\mu\eta_2}{\lambda\eta_1+\mu} b_2 + \frac{\lambda\eta_1\mu\eta_2}{(\lambda\eta_1+\mu)^2} b_1 \\
&= \lambda\eta_1 p_0 + \left[\frac{\mu q_2}{\lambda\eta_1+\mu} + \frac{\lambda\eta_1\mu\eta_2}{(\lambda\eta_1+\mu)^2}\right] b_1 + \frac{\mu\eta_2}{\lambda\eta_1+\mu} b_2 \\
&= \lambda\eta_1 p_0 + \frac{\mu q_2(\lambda\eta_1+\mu) + \lambda\eta_1\mu\eta_2}{(\lambda\eta_1+\mu)^2} b_1 + \frac{\mu\eta_2}{\lambda\eta_1+\mu} b_2 \\
&\Longrightarrow \\
&\frac{\mu\eta_2}{\lambda\eta_1+\mu} b_2 \\
&= \left[1 - \frac{\mu q_2(\lambda\eta_1+\mu) + \lambda\eta_1\mu\eta_2}{(\lambda\eta_1+\mu)^2}\right] b_1 - \lambda\eta_1 p_0 \\
&= \frac{(\lambda\eta_1+\mu)^2 - \mu q_2(\lambda\eta_1+\mu) - \lambda\eta_1\mu\eta_2}{(\lambda\eta_1+\mu)^2} \frac{(\lambda\eta_1+\mu)}{\mu\eta_2} \lambda\eta_1 p_0 \\
&\quad - \lambda\eta_1 p_0 \\
&= \left[\frac{(\lambda\eta_1+\mu)^2 - \mu q_2(\lambda\eta_1+\mu) - \lambda\eta_1\mu\eta_2}{\mu\eta_2(\lambda\eta_1+\mu)} - 1\right] \lambda\eta_1 p_0 \\
&= \frac{(\lambda\eta_1+\mu)^2 - \mu q_2(\lambda\eta_1+\mu) - \lambda\eta_1\mu\eta_2 - \mu\eta_2(\lambda\eta_1+\mu)}{\mu\eta_2(\lambda\eta_1+\mu)} \lambda\eta_1 p_0 \\
&= \frac{(\lambda\eta_1+\mu)[\lambda\eta_1+\mu-\mu q_2-\mu\eta_2] - \lambda\eta_1\mu\eta_2}{\mu\eta_2(\lambda\eta_1+\mu)} \lambda\eta_1 p_0 \\
&= \frac{\lambda\eta_1(\lambda\eta_1+\mu) - \lambda\eta_1\mu\eta_2}{\mu\eta_2(\lambda\eta_1+\mu)} \lambda\eta_1 p_0 \\
&= \frac{\lambda\eta_1(\lambda\eta_1+\mu q_2)}{\mu\eta_2(\lambda\eta_1+\mu)} \lambda\eta_1 p_0 \\
&\Longrightarrow
\end{aligned}
$$

$$
b_2 = \frac{\lambda\eta_1(\lambda\eta_1+\mu q_2)}{(\mu\eta_2)^2} \lambda\eta_1 p_0. \tag{4-86}
$$

From (4-83), (4-84), (4-82), (4-85) and (4-86) and using the Fubini theorem (see Theorem 1.49) we deduce

$$
\begin{aligned}
b_2 &= \frac{\mu q_2}{\lambda\eta_1+\mu} b_2 \\
&\quad + \mu q_2 \lambda\eta_1 \int_0^\infty e^{-(\lambda\eta_1+\mu)x} \int_0^x p_1(\tau) e^{(\lambda\eta_1+\mu)\tau} d\tau dx \\
&\quad + \frac{\mu\eta_2}{\lambda\eta_1+\mu} b_3 \\
&\quad + \mu\eta_2 \lambda\eta_1 \int_0^\infty e^{-(\lambda\eta_1+\mu)x} \int_0^x p_2(\tau) e^{(\lambda\eta_1+\mu)\tau} d\tau dx \\
&= \frac{\mu q_2}{\lambda\eta_1+\mu} b_2 \\
&\quad + \mu q_2 \lambda\eta_1 \int_0^\infty p_1(\tau) e^{(\lambda\eta_1+\mu)\tau} \int_\tau^\infty e^{-(\lambda\eta_1+\mu)x} dx d\tau \\
&\quad + \frac{\mu\eta_2}{\lambda\eta_1+\mu} b_3 \\
&\quad + \mu\eta_2 \lambda\eta_1 \int_0^\infty p_2(\tau) e^{(\lambda\eta_1+\mu)\tau} \int_\tau^\infty e^{-(\lambda\eta_1+\mu)x} dx d\tau \\
&= \frac{\mu q_2}{\lambda\eta_1+\mu} b_2 + \frac{\lambda\eta_1 \mu q_2}{\lambda\eta_1+\mu} \int_0^\infty p_1(\tau) d\tau \\
&\quad + \frac{\mu\eta_2}{\lambda\eta_1+\mu} b_3 + \frac{\mu\eta_2 \lambda\eta_1}{\lambda\eta_1+\mu} \int_0^\infty p_2(\tau) d\tau \\
&= \frac{\mu q_2}{\lambda\eta_1+\mu} b_2 + \frac{\lambda\eta_1 \mu q_2}{\lambda\eta_1+\mu} \int_0^\infty p_1(\tau) d\tau \\
&\quad + \frac{\mu\eta_2}{\lambda\eta_1+\mu} b_3 + \frac{\lambda\eta_1 \mu\eta_2}{(\lambda\eta_1+\mu)^2} b_2 \\
&\quad + \frac{\mu\eta_2 (\lambda\eta_1)^2}{\lambda\eta_1+\mu} \int_0^\infty e^{-(\lambda\eta_1+\mu)\tau} \int_0^\tau p_1(\xi) e^{(\lambda\eta_1+\mu)\xi} d\xi d\tau \\
&= \frac{\mu q_2}{\lambda\eta_1+\mu} b_2 + \frac{\lambda\eta_1 \mu q_2}{\lambda\eta_1+\mu} \int_0^\infty p_1(\tau) d\tau + \frac{\mu\eta_2}{\lambda\eta_1+\mu} b_3 \\
&\quad + \frac{\lambda\eta_1 \mu\eta_2}{(\lambda\eta_1+\mu)^2} b_2 + \frac{\mu\eta_2 (\lambda\eta_1)^2}{(\lambda\eta_1+\mu)^2} \int_0^\infty p_1(\xi) d\xi \\
&= \frac{\mu q_2(\lambda\eta_1+\mu) + \lambda\eta_1 \mu\eta_2}{(\lambda\eta_1+\mu)^2} b_2 \\
&\quad + \left[\frac{\lambda\eta_1 \mu q_2}{(\lambda\eta_1+\mu)^2} + \frac{\mu\eta_2 (\lambda\eta_1)^2}{(\lambda\eta_1+\mu)^3} \right] b_1 + \frac{\mu\eta_2}{\lambda\eta_1+\mu} b_3 \\
&\Longrightarrow \\
&\frac{\mu\eta_2}{\lambda\eta_1+\mu} b_3
\end{aligned}
$$

$$
\begin{aligned}
&= \left[1 - \frac{\mu q_2(\lambda\eta_1 + \mu) + \lambda\eta_1\mu\eta_2}{(\lambda\eta_1 + \mu)^2}\right] b_2 \\
&\quad - \frac{\lambda\eta_1\mu q_2(\lambda\eta_1 + \mu) + \mu\eta_2(\lambda\eta_1)^2}{(\lambda\eta_1 + \mu)^3} b_1 \\
&= \frac{(\lambda\eta_1 + \mu)^2 - \mu q_2(\lambda\eta_1 + \mu) - \lambda\eta_1\mu\eta_2}{(\lambda\eta_1 + \mu)^2} b_2 \\
&\quad - \frac{\lambda\eta_1\mu q_2(\lambda\eta_1 + \mu) + \mu\eta_2(\lambda\eta_1)^2}{(\lambda\eta_1 + \mu)^3} b_1 \\
&= \frac{(\lambda\eta_1 + \mu)(\lambda\eta_1 + \mu\eta_2) - \lambda\eta_1\mu\eta_2}{(\lambda\eta_1 + \mu)^2} \frac{\lambda\eta_1(\lambda\eta_1 + \mu q_2)}{(\mu\eta_2)^2} \lambda\eta_1 p_0 \\
&\quad - \frac{(\lambda\eta_1)^2(\mu q_2 + \mu\eta_2) + \mu\lambda\eta_1\mu q_2}{(\lambda\eta_1 + \mu)^3} \frac{\lambda\eta_1 + \mu}{\mu\eta_2} \lambda\eta_1 p_0 \\
&= \frac{[(\lambda\eta_1 + \mu)(\lambda\eta_1 + \mu\eta_2) - \lambda\eta_1\mu\eta_2]\lambda\eta_1(\lambda\eta_1 + \mu q_2)}{(\mu\eta_2)^2(\lambda\eta_1 + \mu)^2} \lambda\eta_1 p_0 \\
&\quad - \frac{(\lambda\eta_1)^2\mu + \lambda\eta_1\mu^2 q_2}{\mu\eta_2(\lambda\eta_1 + \mu)^2} \lambda\eta_1 p_0 \\
&= \frac{1}{(\mu\eta_2)^2(\lambda\eta_1 + \mu)^2} \lambda\eta_1 p_0 \\
&\quad \times \big\{[(\lambda\eta_1 + \mu)(\lambda\eta_1 + \mu\eta_2) - \lambda\eta_1\mu\eta_2]\lambda\eta_1(\lambda\eta_1 + \mu q_2) \\
&\qquad - \mu^2\eta_2\lambda\eta_1(\lambda\eta_1 + \mu q_2)\big\} \\
&= \frac{\lambda\eta_1(\lambda\eta_1 + \mu q_2)}{(\mu\eta_2)^2(\lambda\eta_1 + \mu)^2} \lambda\eta_1 p_0 \\
&\quad \times \big\{\lambda\eta_1(\lambda\eta_1 + \mu\eta_2) + \mu(\lambda\eta_1 + \mu\eta_2) - \lambda\eta_1\mu\eta_2 - \mu^2\eta_2\big\} \\
&= \frac{\lambda\eta_1(\lambda\eta_1 + \mu q_2)[(\lambda\eta_1)^2 + \lambda\eta_1\mu]}{(\mu\eta_2)^2(\lambda\eta_1 + \mu)^2} \lambda\eta_1 p_0 \\
&= \frac{(\lambda\eta_1)^2(\lambda\eta_1 + \mu q_2)}{(\mu\eta_2)^2(\lambda\eta_1 + \mu)} \lambda\eta_1 p_0 \\
&\Longrightarrow
\end{aligned}
$$

$$
b_3 = \frac{(\lambda\eta_1)^2(\lambda\eta_1 + \mu q_2)}{(\mu\eta_2)^3} \lambda\eta_1 p_0. \tag{4-87}
$$

Through combining (4-85), (4-86) and (4-87) with (4-84) and (4-82) we get

$$
\begin{aligned}
b_3 &= \frac{\mu q_2}{\lambda\eta_1 + \mu} b_3 + \frac{\lambda\eta_1\mu q_2}{(\lambda\eta_1 + \mu)^2} b_2 + \frac{(\lambda\eta_1)^2\mu q_2}{(\lambda\eta_1 + \mu)^3} b_1 \\
&\quad + \frac{\mu\eta_2}{\lambda\eta_1 + \mu} b_4 + \frac{\lambda\eta_1\mu\eta_2}{(\lambda\eta_1 + \mu)^2} b_3 + \frac{(\lambda\eta_1)^2\mu\eta_2}{(\lambda\eta_1 + \mu)^3} b_2 \\
&\quad + \frac{(\lambda\eta_1)^3\mu\eta_2}{(\lambda\eta_1 + \mu)^4} b_1
\end{aligned}
$$

$$
\begin{aligned}
&\Longrightarrow \\
&\frac{\mu\eta_2}{(\lambda\eta_1+\mu)} b_4 \\
&= \left[1 - \frac{\mu q_2}{\lambda\eta_1+\mu} - \frac{\lambda\eta_1\mu\eta_2}{(\lambda\eta_1+\mu)^2}\right] b_3 \\
&\quad - \left[\frac{\lambda\eta_1\mu q_2}{(\lambda\eta_1+\mu)^2} + \frac{(\lambda\eta_1)^2\mu\eta_2}{(\lambda\eta_1+\mu)^3}\right] b_2 \\
&\quad - \left[\frac{(\lambda\eta_1)^2\mu q_2}{(\lambda\eta_1+\mu)^3} + \frac{(\lambda\eta_1)^3\mu\eta_2}{(\lambda\eta_1+\mu)^4}\right] b_1 \\
&= \frac{(\lambda\eta_1+\mu)(\lambda\eta_1+\mu\eta_2)-\lambda\eta_1\mu\eta_2}{(\lambda\eta_1+\mu)^2} b_3 \\
&\quad - \frac{\lambda\eta_1\mu q_2(\lambda\eta_1+\mu)+(\lambda\eta_1)^2\mu\eta_2}{(\lambda\eta_1+\mu)^3} b_2 \\
&\quad - \frac{\mu q_2(\lambda\eta_1)^2(\lambda\eta_1+\mu)+\mu\eta_2(\lambda\eta_1)^3}{(\lambda\eta_1+\mu)^4} b_1 \\
&= \frac{(\lambda\eta_1+\mu)(\lambda\eta_1+\mu\eta_2)-\lambda\eta_1\mu\eta_2}{(\lambda\eta_1+\mu)^2} \frac{(\lambda\eta_1)^2(\lambda\eta_1+\mu q_2)}{(\mu\eta_2)^3}\lambda\eta_1 p_0 \\
&\quad - \frac{(\lambda\eta_1)^2\mu+\lambda\eta_1\mu^2 q_2}{(\lambda\eta_1+\mu)^3} \frac{\lambda\eta_1(\lambda\eta_1+\mu q_2)}{(\mu\eta_2)^2}\lambda\eta_1 p_0 \\
&\quad - \frac{(\lambda\eta_1)^3\mu+(\lambda\eta_1)^2\mu^2 q_2}{(\lambda\eta_1+\mu)^4} \frac{\lambda\eta_1+\mu}{\mu\eta_2}\lambda\eta_1 p_0 \\
&= \frac{(\lambda\eta_1+\mu)(\lambda\eta_1+\mu\eta_2)-\lambda\eta_1\mu\eta_2}{(\lambda\eta_1+\mu)^2} \frac{(\lambda\eta_1)^2(\lambda\eta_1+\mu q_2)}{(\mu\eta_2)^3}\lambda\eta_1 p_0 \\
&\quad - \frac{(\lambda\eta_1)^2\mu(\lambda\eta_1+\mu q_2)^2}{(\mu\eta_2)^2(\lambda\eta_1+\mu)^3}\lambda\eta_1 p_0 \\
&\quad - \frac{(\lambda\eta_1)^2\mu(\lambda\eta_1+\mu q_2)}{(\lambda\eta_1+\mu)^3\mu\eta_2}\lambda\eta_1 p_0 \\
&= \frac{(\lambda\eta_1+\mu)(\lambda\eta_1+\mu\eta_2)-\lambda\eta_1\mu\eta_2}{(\lambda\eta_1+\mu)^2} \frac{(\lambda\eta_1)^2(\lambda\eta_1+\mu q_2)}{(\mu\eta_2)^3}\lambda\eta_1 p_0 \\
&\quad - \frac{(\lambda\eta_1)^2\mu(\lambda\eta_1+\mu q_2)[(\lambda\eta_1+\mu q_2)+\mu\eta_2]}{(\mu\eta_2)^2(\lambda\eta_1+\mu)^3}\lambda\eta_1 p_0 \\
&= \frac{(\lambda\eta_1+\mu)(\lambda\eta_1+\mu\eta_2)-\lambda\eta_1\mu\eta_2}{(\lambda\eta_1+\mu)^2} \frac{(\lambda\eta_1)^2(\lambda\eta_1+\mu q_2)}{(\mu\eta_2)^3}\lambda\eta_1 p_0 \\
&\quad - \frac{(\lambda\eta_1)^2\mu(\lambda\eta_1+\mu q_2)(\lambda\eta_1+\mu)}{(\mu\eta_2)^2(\lambda\eta_1+\mu)^3}\lambda\eta_1 p_0 \\
&= \frac{(\lambda\eta_1+\mu)(\lambda\eta_1+\mu\eta_2)-\lambda\eta_1\mu\eta_2}{(\lambda\eta_1+\mu)^2} \frac{(\lambda\eta_1)^2(\lambda\eta_1+\mu q_2)}{(\mu\eta_2)^3}\lambda\eta_1 p_0
\end{aligned}
$$

$$
\begin{aligned}
&\quad - \frac{(\lambda\eta_1)^2\mu(\lambda\eta_1+\mu q_2)}{(\mu\eta_2)^2(\lambda\eta_1+\mu)^2}\lambda\eta_1 p_0 \\
&= \frac{(\lambda\eta_1)^2(\lambda\eta_1+\mu q_2)}{(\mu\eta_2)^3(\lambda\eta_1+\mu)^2}\lambda\eta_1 p_0 \\
&\quad \times \left[(\lambda\eta_1)^2+\lambda\eta_1\mu\eta_2+\lambda\eta_1\mu+\mu^2\eta_2-\lambda\eta_1\mu\eta_2-\mu^2\eta_2\right] \\
&= \frac{(\lambda\eta_1)^3(\lambda\eta_1+\mu q_2)}{(\mu\eta_2)^3(\lambda\eta_1+\mu)}\lambda\eta_1 p_0 \\
&\Longrightarrow \\
b_4 &= \frac{(\lambda\eta_1)^3(\lambda\eta_1+\mu q_2)}{(\mu\eta_2)^4}\lambda\eta_1 p_0. \qquad (4\text{-}88)
\end{aligned}
$$

Similar to (4-86), (4-87) and (4-88), by induction we calculate

$$
\begin{aligned}
b_n = &\frac{\mu q_2}{\lambda\eta_1+\mu}b_n + \frac{\mu q_2\lambda\eta_1}{(\lambda\eta_1+\mu)^2}b_{n-1} + \frac{\mu q_2(\lambda\eta_1)^2}{(\lambda\eta_1+\mu)^3}b_{n-2} \\
&+\cdots\cdots+\frac{\mu q_2(\lambda\eta_1)^{n-1}}{(\lambda\eta_1+\mu)^n}b_1 \\
&+\frac{\mu\eta_2}{\lambda\eta_1+\mu}b_{n+1}+\frac{\mu\eta_2\lambda\eta_1}{(\lambda\eta_1+\mu)^2}b_n+\frac{\mu\eta_2(\lambda\eta_1)^2}{(\lambda\eta_1+\mu)^3}b_{n-1} \\
&+\cdots\cdots+\frac{\mu\eta_2(\lambda\eta_1)^n}{(\lambda\eta_1+\mu)^{n+1}}b_1, \quad n \geq 1. \qquad (4\text{-}89)
\end{aligned}
$$

By combining (4-86), (4-87) and (4-88) with (4-89) we obtain

$$
b_n = \frac{(\lambda\eta_1)^{n-1}(\lambda\eta_1+\mu q_2)}{(\mu\eta_2)^n}\lambda\eta_1 p_0, \quad n \geq 2. \qquad (4\text{-}90)
$$

In the following we prove (4-90) by using mathematical induction. (4-86), (4-87) and (4-88) show that (4-90) holds for $k = 2, 3, 4$. Assume that (4-90) holds for $k \leq n$, i.e., $b_k = \frac{(\lambda\eta_1)^{k-1}(\lambda\eta_1+\mu q_2)}{(\mu\eta_2)^k}\lambda\eta_1 p_0,\ k \leq n$. We will verify that (4-90) holds for $k = n+1$.

By (4-89) and $q_2 + \eta_2 = 1$ we derive, by noting an inductive assumption,

$$
\begin{aligned}
b_n = &\frac{\mu q_2}{\lambda\eta_1+\mu}b_n + \frac{\mu q_2\lambda\eta_1}{(\lambda\eta_1+\mu)^2}b_{n-1} + \frac{\mu q_2(\lambda\eta_1)^2}{(\lambda\eta_1+\mu)^3}b_{n-2} \\
&+\cdots\cdots+\frac{\mu q_2(\lambda\eta_1)^{n-1}}{(\lambda\eta_1+\mu)^n}b_1 \\
&+\frac{\mu\eta_2}{\lambda\eta_1+\mu}b_{n+1}+\frac{\mu\eta_2\lambda\eta_1}{(\lambda\eta_1+\mu)^2}b_n+\frac{\mu\eta_2(\lambda\eta_1)^2}{(\lambda\eta_1+\mu)^3}b_{n-1} \\
&+\cdots\cdots+\frac{\mu\eta_2(\lambda\eta_1)^{n-1}}{(\lambda\eta_1+\mu)^n}b_2+\frac{\mu\eta_2(\lambda\eta_1)^n}{(\lambda\eta_1+\mu)^{n+1}}b_1
\end{aligned}
$$

$$
\begin{aligned}
&\Longrightarrow \\
&\frac{\mu\eta_2}{\lambda\eta_1+\mu}b_{n+1} \\
&= \left[1-\frac{\mu q_2}{\lambda\eta_1+\mu}-\frac{\mu\eta_2\lambda\eta_1}{(\lambda\eta_1+\mu)^2}\right]b_n \\
&\quad -\left[\frac{\mu q_2\lambda\eta_1}{(\lambda\eta_1+\mu)^2}+\frac{\mu\eta_2(\lambda\eta_1)^2}{(\lambda\eta_1+\mu)^3}\right]b_{n-1} \\
&\quad -\left[\frac{\mu q_2(\lambda\eta_1)^2}{(\lambda\eta_1+\mu)^3}+\frac{\mu\eta_2(\lambda\eta_1)^3}{(\lambda\eta_1+\mu)^4}\right]b_{n-2} \\
&\quad -\left[\frac{\mu q_2(\lambda\eta_1)^3}{(\lambda\eta_1+\mu)^4}+\frac{\mu\eta_2(\lambda\eta_1)^4}{(\lambda\eta_1+\mu)^5}\right]b_{n-3} \\
&\quad -\cdots-\left[\frac{\mu q_2(\lambda\eta_1)^{n-2}}{(\lambda\eta_1+\mu)^{n-1}}+\frac{\mu\eta_2(\lambda\eta_1)^{n-1}}{(\lambda\eta_1+\mu)^n}\right]b_2 \\
&\quad -\left[\frac{\mu q_2(\lambda\eta_1)^{n-1}}{(\lambda\eta_1+\mu)^n}+\frac{\mu\eta_2(\lambda\eta_1)^n}{(\lambda\eta_1+\mu)^{n+1}}\right]b_1 \\
&= \frac{(\lambda\eta_1+\mu)^2-\mu q_2(\lambda\eta_1+\mu)-\lambda\eta_1\mu\eta_2}{(\lambda\eta_1+\mu)^2}b_n \\
&\quad -\frac{\mu q_2\lambda\eta_1(\lambda\eta_1+\mu)+\mu\eta_2(\lambda\eta_1)^2}{(\lambda\eta_1+\mu)^3}b_{n-1} \\
&\quad -\frac{\mu q_2(\lambda\eta_1)^2(\lambda\eta_1+\mu)+\mu\eta_2(\lambda\eta_1)^3}{(\lambda\eta_1+\mu)^4}b_{n-2} \\
&\quad -\frac{\mu q_2(\lambda\eta_1)^3(\lambda\eta_1+\mu)+\mu\eta_2(\lambda\eta_1)^4}{(\lambda\eta_1+\mu)^5}b_{n-3} \\
&\quad -\cdots-\frac{\mu q_2(\lambda\eta_1)^{n-2}(\lambda\eta_1+\mu)+\mu\eta_2(\lambda\eta_1)^{n-1}}{(\lambda\eta_1+\mu)^n}b_2 \\
&\quad -\frac{\mu q_2(\lambda\eta_1)^{n-1}(\lambda\eta_1+\mu)+\mu\eta_2(\lambda\eta_1)^n}{(\lambda\eta_1+\mu)^{n+1}}b_1 \\
&= \frac{(\lambda\eta_1+\mu)(\lambda\eta_1+\mu\eta_2)-\lambda\eta_1\mu\eta_2}{(\lambda\eta_1+\mu)^2}b_n \\
&\quad -\frac{(\lambda\eta_1)^2(\mu q_2+\mu\eta_2)+\lambda\eta_1\mu^2q_2}{(\lambda\eta_1+\mu)^3}b_{n-1} \\
&\quad -\frac{(\lambda\eta_1)^3(\mu q_2+\mu\eta_2)+(\lambda\eta_1)^2\mu^2q_2}{(\lambda\eta_1+\mu)^4}b_{n-2} \\
&\quad -\frac{(\lambda\eta_1)^4(\mu q_2+\mu\eta_2)+(\lambda\eta_1)^3\mu^2q_2}{(\lambda\eta_1+\mu)^5}b_{n-3} \\
&\quad -\cdots-\frac{(\lambda\eta_1)^{n-1}(\mu q_2+\mu\eta_2)+(\lambda\eta_1)^{n-2}\mu^2q_2}{(\lambda\eta_1+\mu)^n}b_2
\end{aligned}
$$

$$
\begin{aligned}
&- \frac{(\lambda\eta_1)^n(\mu q_2 + \mu\eta_2) + (\lambda\eta_1)^{n-1}\mu^2 q_2}{(\lambda\eta_1 + \mu)^{n+1}} b_1 \\
= &\ \frac{\lambda\eta_1(\lambda\eta_1 + \mu) + \mu\eta_2(\lambda\eta_1 + \mu) - \lambda\eta_1\mu\eta_2}{(\lambda\eta_1 + \mu)^2} \\
&\quad \times \frac{(\lambda\eta_1)^{n-1}(\lambda\eta_1 + \mu q_2)}{(\mu\eta_2)^n} \lambda\eta_1 p_0 \\
&- \frac{\lambda\eta_1\mu(\lambda\eta_1 + \mu q_2)}{(\lambda\eta_1 + \mu)^3} \frac{(\lambda\eta_1)^{n-2}(\lambda\eta_1 + \mu q_2)}{(\mu\eta_2)^{n-1}} \lambda\eta_1 p_0 \\
&- \frac{(\lambda\eta_1)^2\mu(\lambda\eta_1 + \mu q_2)}{(\lambda\eta_1 + \mu)^4} \frac{(\lambda\eta_1)^{n-3}(\lambda\eta_1 + \mu q_2)}{(\mu\eta_2)^{n-2}} \lambda\eta_1 p_0 \\
&- \frac{(\lambda\eta_1)^3\mu(\lambda\eta_1 + \mu q_2)}{(\lambda\eta_1 + \mu)^5} \frac{(\lambda\eta_1)^{n-4}(\lambda\eta_1 + \mu q_2)}{(\mu\eta_2)^{n-3}} \lambda\eta_1 p_0 \\
&- \cdots - \frac{(\lambda\eta_1)^{n-2}\mu(\lambda\eta_1 + \mu q_2)}{(\lambda\eta_1 + \mu)^n} \frac{\lambda\eta_1(\lambda\eta_1 + \mu q_2)}{(\mu\eta_2)^2} \lambda\eta_1 p_0 \\
&- \frac{(\lambda\eta_1)^{n-1}\mu(\lambda\eta_1 + \mu q_2)}{(\lambda\eta_1 + \mu)^{n+1}} \frac{\lambda\eta_1 + \mu}{\mu\eta_2} \lambda\eta_1 p_0 \\
= &\ \frac{[\lambda\eta_1(\lambda\eta_1 + \mu) + \mu^2\eta_2](\lambda\eta_1)^{n-1}(\lambda\eta_1 + \mu q_2)}{(\lambda\eta_1 + \mu)^2(\mu\eta_2)^n} \lambda\eta_1 p_0 \\
&- \frac{\mu(\lambda\eta_1)^{n-1}(\lambda\eta_1 + \mu q_2)^2}{(\lambda\eta_1 + \mu)^3(\mu\eta_2)^{n-1}} \lambda\eta_1 p_0 \\
&- \frac{\mu(\lambda\eta_1)^{n-1}(\lambda\eta_1 + \mu q_2)^2}{(\lambda\eta_1 + \mu)^4(\mu\eta_2)^{n-2}} \lambda\eta_1 p_0 \\
&- \frac{\mu(\lambda\eta_1)^{n-1}(\lambda\eta_1 + \mu q_2)^2}{(\lambda\eta_1 + \mu)^5(\mu\eta_2)^{n-3}} \lambda\eta_1 p_0 \\
&- \cdots - \frac{\mu(\lambda\eta_1)^{n-1}(\lambda\eta_1 + \mu q_2)^2}{(\lambda\eta_1 + \mu)^n(\mu\eta_2)^2} \lambda\eta_1 p_0 \\
&- \frac{\mu(\lambda\eta_1)^{n-1}(\lambda\eta_1 + \mu q_2)}{(\lambda\eta_1 + \mu)^n\mu\eta_2} \lambda\eta_1 p_0 \\
= &\ \frac{[\lambda\eta_1(\lambda\eta_1 + \mu) + \mu^2\eta_2](\lambda\eta_1)^{n-1}(\lambda\eta_1 + \mu q_2)}{(\lambda\eta_1 + \mu)^2(\mu\eta_2)^n} \lambda\eta_1 p_0 \\
&- \mu(\lambda\eta_1)^{n-1}(\lambda\eta_1 + \mu q_2)^2 \left[\frac{1}{(\lambda\eta_1 + \mu)^3(\mu\eta_2)^{n-1}} \right. \\
&\left. + \frac{1}{(\lambda\eta_1 + \mu)^4(\mu\eta_2)^{n-2}} + \cdots + \frac{1}{(\lambda\eta_1 + \mu)^n(\mu\eta_2)^2} \right] \lambda\eta_1 p_0 \\
&- \frac{\mu(\lambda\eta_1)^{n-1}(\lambda\eta_1 + \mu q_2)}{(\lambda\eta_1 + \mu)^n\mu\eta_2} \lambda\eta_1 p_0
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{[\lambda\eta_1(\lambda\eta_1+\mu)+\mu^2\eta_2](\lambda\eta_1)^{n-1}(\lambda\eta_1+\mu q_2)}{(\lambda\eta_1+\mu)^2(\mu\eta_2)^n}\lambda\eta_1 p_0 \\
&\quad - \mu(\lambda\eta_1)^{n-1}(\lambda\eta_1+\mu q_2)^2 \frac{\frac{1}{(\lambda\eta_1+\mu)^3(\mu\eta_2)^{n-1}}}{1-\frac{\mu\eta_2}{\lambda\eta_1+\mu}} \\
&\quad\times\left[1-\left(\frac{\mu\eta_2}{\lambda\eta_1+\mu}\right)^{n-2}\right]\lambda\eta_1 p_0 \\
&\quad - \frac{\mu(\lambda\eta_1)^{n-1}(\lambda\eta_1+\mu q_2)}{(\lambda\eta_1+\mu)^n\mu\eta_2}\lambda\eta_1 p_0 \\
&= \frac{[\lambda\eta_1(\lambda\eta_1+\mu)+\mu^2\eta_2](\lambda\eta_1)^{n-1}(\lambda\eta_1+\mu q_2)}{(\lambda\eta_1+\mu)^2(\mu\eta_2)^n}\lambda\eta_1 p_0 \\
&\quad - \frac{\mu(\lambda\eta_1)^{n-1}(\lambda\eta_1+\mu q_2)^2}{(\lambda\eta_1+\mu)^2(\mu\eta_2)^{n-1}(\lambda\eta_1+\mu q_2)} \\
&\quad\times\left[1-\left(\frac{\mu\eta_2}{\lambda\eta_1+\mu}\right)^{n-2}\right]\lambda\eta_1 p_0 \\
&\quad - \frac{\mu(\lambda\eta_1)^{n-1}(\lambda\eta_1+\mu q_2)}{(\lambda\eta_1+\mu)^n\mu\eta_2}\lambda\eta_1 p_0 \\
&= \frac{[\lambda\eta_1(\lambda\eta_1+\mu)+\mu^2\eta_2](\lambda\eta_1)^{n-1}(\lambda\eta_1+\mu q_2)}{(\lambda\eta_1+\mu)^2(\mu\eta_2)^n}\lambda\eta_1 p_0 \\
&\quad - \frac{\mu(\lambda\eta_1)^{n-1}(\lambda\eta_1+\mu q_2)[(\lambda\eta_1+\mu)^{n-2}-(\mu\eta_2)^{n-2}]}{(\lambda\eta_1+\mu)^n(\mu\eta_2)^{n-1}}\lambda\eta_1 p_0 \\
&\quad - \frac{\mu(\lambda\eta_1)^{n-1}(\lambda\eta_1+\mu q_2)}{(\lambda\eta_1+\mu)^n\mu\eta_2}\lambda\eta_1 p_0 \\
&= \frac{[\lambda\eta_1(\lambda\eta_1+\mu)+\mu^2\eta_2](\lambda\eta_1)^{n-1}(\lambda\eta_1+\mu q_2)}{(\lambda\eta_1+\mu)^2(\mu\eta_2)^n}\lambda\eta_1 p_0 \\
&\quad - \frac{\mu(\lambda\eta_1)^{n-1}(\lambda\eta_1+\mu q_2)}{(\lambda\eta_1+\mu)^n(\mu\eta_2)^{n-1}}\lambda\eta_1 p_0 \\
&\quad\times\left[(\lambda\eta_1+\mu)^{n-2}-(\mu\eta_2)^{n-2}+(\mu\eta_2)^{n-2}\right] \\
&= \frac{[\lambda\eta_1(\lambda\eta_1+\mu)+\mu^2\eta_2](\lambda\eta_1)^{n-1}(\lambda\eta_1+\mu q_2)}{(\lambda\eta_1+\mu)^2(\mu\eta_2)^n}\lambda\eta_1 p_0 \\
&\quad - \frac{\mu(\lambda\eta_1)^{n-1}(\lambda\eta_1+\mu q_2)(\lambda\eta_1+\mu)^{n-2}}{(\lambda\eta_1+\mu)^n(\mu\eta_2)^{n-1}}\lambda\eta_1 p_0 \\
&= \frac{[\lambda\eta_1(\lambda\eta_1+\mu)+\mu^2\eta_2](\lambda\eta_1)^{n-1}(\lambda\eta_1+\mu q_2)}{(\lambda\eta_1+\mu)^2(\mu\eta_2)^n}\lambda\eta_1 p_0 \\
&\quad - \frac{\mu(\lambda\eta_1)^{n-1}(\lambda\eta_1+\mu q_2)}{(\lambda\eta_1+\mu)^2(\mu\eta_2)^{n-1}}\lambda\eta_1 p_0
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{(\lambda\eta_1)^{n-1}(\lambda\eta_1+\mu q_2)[\lambda\eta_1(\lambda\eta_1+\mu)+\mu^2\eta_2-\mu^2\eta_2]}{(\lambda\eta_1+\mu)^2(\mu\eta_2)^n}\lambda\eta_1 p_0 \\
&= \frac{(\lambda\eta_1)^{n-1}(\lambda\eta_1+\mu q_2)\lambda\eta_1(\lambda\eta_1+\mu)}{(\lambda\eta_1+\mu)^2(\mu\eta_2)^n}\lambda\eta_1 p_0 \\
&= \frac{(\lambda\eta_1)^{n}(\lambda\eta_1+\mu q_2)}{(\lambda\eta_1+\mu)(\mu\eta_2)^n}\lambda\eta_1 p_0 \\
&\Longrightarrow \\
b_{n+1} &= \frac{(\lambda\eta_1)^{n}(\lambda\eta_1+\mu q_2)}{(\mu\eta_2)^{n+1}}\lambda\eta_1 p_0.
\end{aligned}
$$

This shows that (4-90) holds for $k = n+1$.

From (4-84) and the Fubini theorem (see Theorem 1.49) we estimate

$$
\begin{aligned}
&\|p_n\|_{L^1[0,\infty)} = \int_0^\infty |p_n(x)|dx \\
&\le \int_0^\infty |b_n| e^{-(\lambda\eta_1+\mu)x}dx \\
&\quad + \lambda\eta_1 \int_0^\infty e^{-(\lambda\eta_1+\mu)x}\int_0^x |p_{n-1}(\tau)|e^{(\lambda\eta_1+\mu)\tau}d\tau dx \\
&= \frac{1}{\lambda\eta_1+\mu}|b_n| \\
&\quad + \lambda\eta_1 \int_0^\infty |p_{n-1}(\tau)|e^{(\lambda\eta_1+\mu)\tau}\int_\tau^\infty e^{-(\lambda\eta_1+\mu)x}dx d\tau \\
&= \frac{1}{\lambda\eta_1+\mu}|b_n| + \frac{\lambda\eta_1}{\lambda\eta_1+\mu}\int_0^\infty |p_{n-1}(\tau)|d\tau \\
&\le \frac{1}{\lambda\eta_1+\mu}|b_n| + \frac{\lambda\eta_1}{\lambda\eta_1+\mu}\int_0^\infty |b_{n-1}|e^{-(\lambda\eta_1+\mu)\tau}d\tau \\
&\quad + \frac{(\lambda\eta_1)^2}{\lambda\eta_1+\mu}\int_0^\infty e^{-(\lambda\eta_1+\mu)\tau}\int_0^\tau |p_{n-2}(\xi)|e^{(\lambda\eta_1+\mu)\xi}d\xi d\tau \\
&= \frac{1}{\lambda\eta_1+\mu}|b_n| + \frac{\lambda\eta_1}{(\lambda\eta_1+\mu)^2}|b_{n-1}| \\
&\quad + \frac{(\lambda\eta_1)^2}{\lambda\eta_1+\mu}\int_0^\infty |p_{n-2}(\xi)|e^{(\lambda\eta_1+\mu)\xi}\int_\xi^\infty e^{-(\lambda\eta_1+\mu)\tau}d\tau d\xi \\
&= \frac{1}{\lambda\eta_1+\mu}|b_n| + \frac{\lambda\eta_1}{(\lambda\eta_1+\mu)^2}|b_{n-1}| \\
&\quad + \frac{(\lambda\eta_1)^2}{(\lambda\eta_1+\mu)^2}\int_0^\infty |p_{n-2}(\xi)|d\xi \\
&\le \frac{1}{\lambda\eta_1+\mu}|b_n| + \frac{\lambda\eta_1}{(\lambda\eta_1+\mu)^2}|b_{n-1}| + \frac{(\lambda\eta_1)^2}{(\lambda\eta_1+\mu)^3}|b_{n-2}|
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{(\lambda\eta_1)^3}{(\lambda\eta_1+\mu)^2}\int_0^\infty e^{-(\lambda\eta_1+\mu)\xi}\int_0^\xi |p_{n-3}(\tau)|e^{(\lambda\eta_1+\mu)\tau}d\tau d\xi \\
\le{}& \cdots \\
\le{}& \frac{1}{\lambda\eta_1+\mu}|b_n| + \frac{\lambda\eta_1}{(\lambda\eta_1+\mu)^2}|b_{n-1}| + \frac{(\lambda\eta_1)^2}{(\lambda\eta_1+\mu)^3}|b_{n-2}| \\
&+\cdots+\frac{(\lambda\eta_1)^{n-1}}{(\lambda\eta_1+\mu)^n}|b_1| \\
={}& \frac{1}{\lambda\eta_1+\mu}\frac{(\lambda\eta_1)^{n-1}(\lambda\eta_1+\mu q_2)}{(\mu\eta_2)^n}\lambda\eta_1|p_0| \\
&+\frac{\lambda\eta_1}{(\lambda\eta_1+\mu)^2}\frac{(\lambda\eta_1)^{n-2}(\lambda\eta_1+\mu q_2)}{(\mu\eta_2)^{n-1}}\lambda\eta_1|p_0| \\
&+\frac{(\lambda\eta_1)^2}{(\lambda\eta_1+\mu)^3}\frac{(\lambda\eta_1)^{n-3}(\lambda\eta_1+\mu q_2)}{(\mu\eta_2)^{n-2}}\lambda\eta_1|p_0| \\
&+\frac{(\lambda\eta_1)^3}{(\lambda\eta_1+\mu)^4}\frac{(\lambda\eta_1)^{n-4}(\lambda\eta_1+\mu q_2)}{(\mu\eta_2)^{n-3}}\lambda\eta_1|p_0| \\
&+\cdots \\
&+\frac{(\lambda\eta_1)^{n-2}}{(\lambda\eta_1+\mu)^{n-1}}\frac{\lambda\eta_1(\lambda\eta_1+\mu q_2)}{(\mu\eta_2)^2}\lambda\eta_1|p_0| \\
&+\frac{(\lambda\eta_1)^{n-1}}{(\lambda\eta_1+\mu)^n}\frac{\lambda\eta_1+\mu}{\mu\eta_2}\lambda\eta_1|p_0| \\
={}& (\lambda\eta_1+\mu q_2)|p_0|\left\{\frac{(\lambda\eta_1)^n}{(\lambda\eta_1+\mu)(\mu\eta_2)^n}+\frac{(\lambda\eta_1)^n}{(\lambda\eta_1+\mu)^2(\mu\eta_2)^{n-1}}\right. \\
&+\frac{(\lambda\eta_1)^n}{(\lambda\eta_1+\mu)^3(\mu\eta_2)^{n-2}}+\frac{(\lambda\eta_1)^n}{(\lambda\eta_1+\mu)^4(\mu\eta_2)^{n-3}} \\
&+\cdots \\
&\left.+\frac{(\lambda\eta_1)^n}{(\lambda\eta_1+\mu)^{n-1}(\mu\eta_2)^2}\right\}+\frac{(\lambda\eta_1)^n}{(\lambda\eta_1+\mu)^{n-1}\mu\eta_2}|p_0| \\
={}& (\lambda\eta_1)^n(\lambda\eta_1+\mu q_2)|p_0|\left\{\frac{1}{(\lambda\eta_1+\mu)(\mu\eta_2)^n}\right. \\
&+\frac{1}{(\lambda\eta_1+\mu)^2(\mu\eta_2)^{n-1}}+\frac{1}{(\lambda\eta_1+\mu)^3(\mu\eta_2)^{n-2}} \\
&\left.+\cdots+\frac{1}{(\lambda\eta_1+\mu)^{n-1}(\mu\eta_2)^2}\right\} \\
&+\frac{(\lambda\eta_1)^n}{(\lambda\eta_1+\mu)^{n-1}\mu\eta_2}|p_0| \\
={}& (\lambda\eta_1)^n(\lambda\eta_1+\mu q_2)\frac{\frac{1}{(\lambda\eta_1+\mu)(\mu\eta_2)^n}}{1-\frac{\mu\eta_2}{\lambda\eta_1+\mu}}\left[1-\left(\frac{\mu\eta_2}{\lambda\eta_1+\mu}\right)^{n-1}\right]|p_0|
\end{aligned}
$$

$$
\begin{aligned}
&\quad + \frac{(\lambda\eta_1)^n}{(\lambda\eta_1+\mu)^{n-1}\mu\eta_2}|p_0| \\
&= (\lambda\eta_1)^n(\lambda\eta_1+\mu q_2)|p_0|\frac{1}{(\lambda\eta_1+\mu q_2)(\mu\eta_2)^n} \\
&\quad \times \left[1-\left(\frac{\mu\eta_2}{\lambda\eta_1+\mu}\right)^{n-1}\right] \\
&\quad + \frac{(\lambda\eta_1)^n}{(\lambda\eta_1+\mu)^{n-1}\mu\eta_2}|p_0| \\
&= \left(\frac{\lambda\eta_1}{\mu\eta_2}\right)^n \left[1-\left(\frac{\mu\eta_2}{\lambda\eta_1+\mu}\right)^{n-1}\right]|p_0| \\
&\quad + \frac{(\lambda\eta_1)^n}{(\lambda\eta_1+\mu)^{n-1}\mu\eta_2}|p_0| \\
&= \left(\frac{\lambda\eta_1}{\mu\eta_2}\right)^n |p_0| - \left(\frac{\lambda\eta_1}{\mu\eta_2}\right)^n \frac{(\mu\eta_2)^{n-1}}{(\lambda\eta_1+\mu)^{n-1}}|p_0| \\
&\quad + \frac{(\lambda\eta_1)^n}{(\lambda\eta_1+\mu)^{n-1}\mu\eta_2}|p_0| \\
&= \left(\frac{\lambda\eta_1}{\mu\eta_2}\right)^n |p_0| - \frac{(\lambda\eta_1)^n}{\mu\eta_2(\lambda\eta_1+\mu)^{n-1}}|p_0| \\
&\quad + \frac{(\lambda\eta_1)^n}{\mu\eta_2(\lambda\eta_1+\mu)^{n-1}}|p_0| \\
&= \left(\frac{\lambda\eta_1}{\mu\eta_2}\right)^n |p_0|, \quad n \geq 1.
\end{aligned} \tag{4-91}
$$

This implies

$$
\begin{aligned}
\|p\| &= |p_0| + \sum_{n=1}^{\infty} \|p_n\|_{L^1[0,\infty)} \\
&\leq |p_0| + \sum_{n=1}^{\infty} \left(\frac{\lambda\eta_1}{\mu\eta_2}\right)^n |p_0| \\
&= |p_0| \left[1 + \sum_{n=1}^{\infty} \left(\frac{\lambda\eta_1}{\mu\eta_2}\right)^n\right] \\
&= |p_0| \sum_{n=0}^{\infty} \left(\frac{\lambda\eta_1}{\mu\eta_2}\right)^n \\
&= \frac{\mu\eta_2}{\mu\eta_2-\lambda\eta_1}|p_0| < \infty,
\end{aligned} \tag{4-92}
$$

which shows that 0 is an eigenvalue of $A+U+E$ (see Definition 1.22). By (4-83)–(4-85), (4-90) and (4-92) we know that $p(x) = (p_0, p_1(x), p_2(x), \dots)$, where

$$p_1(x) = b_1 e^{-(\lambda\eta_1+\mu)x}, \tag{4-93}$$

$$p_n(x) = e^{-(\lambda\eta_1+\mu)x} \sum_{k=0}^{n-1} \frac{(\lambda\eta_1 x)^k}{k!} a_{n-k}, \quad n \geq 2, \tag{4-94}$$

$$b_1 = \frac{\lambda\eta_1+\mu}{\mu\eta_2} p_0, \quad b_n = \left(\frac{\lambda\eta_1}{\mu\eta_2}\right)^n (\lambda\eta_1 + \mu q_2) p_0, \ n \geq 2 \tag{4-95}$$

are eigenvectors of $A+U+E$ corresponding to 0. Moreover, it is easy to see from (4-93)–(4-95) that the space spanned by the eigenvectors corresponding to 0 is a one-dimensional linear space, i.e., the geometric multiplicity of 0 is 1 (see Definition 1.23). □

Lemma 4.7. *If* $\lambda\eta_1 < \mu\eta_2$, *then* $2\sqrt{\lambda\eta_1\mu\eta_2} - \lambda\eta_1 - \mu\eta_2$ *is an eigenvalue of* $A+U+E$ *with geometric multiplicity* 1.

Proof. We consider the equation

$$\left[(A+U+E) - \left(2\sqrt{\lambda\eta_1\mu\eta_2} - \lambda\eta_1 - \mu\eta_2\right) I\right] p = 0,$$

i.e.,

$$\left(2\sqrt{\lambda\eta_1\mu\eta_2} - \mu\eta_2\right) p_0 = \mu\eta_2 \int_0^\infty p_1(x)dx, \tag{4-96}$$

$$\frac{dp_1(x)}{dx} = -\left(2\sqrt{\lambda\eta_1\mu\eta_2} + \mu - \mu\eta_2\right) p_1(x), \tag{4-97}$$

$$\begin{aligned}\frac{dp_n(x)}{dx} &= -\left(2\sqrt{\lambda\eta_1\mu\eta_2} + \mu - \mu\eta_2\right) p_n(x) \\ &\quad + \lambda\eta_1 p_{n-1}(x), \ n \geq 2,\end{aligned} \tag{4-98}$$

$$p_1(0) = \lambda\eta_1 p_0 + \mu q_2 \int_0^\infty p_1(x)dx + \mu\eta_2 \int_0^\infty p_2(x)dx, \tag{4-99}$$

$$p_n(0) = \mu q_2 \int_0^\infty p_n(x)dx + \mu\eta_2 \int_0^\infty p_{n+1}(x)dx, \ n \geq 2. \tag{4-100}$$

Solving (4-97) and (4-98) and using the relation $\eta_2 = 1 - q_2$ we have

$$p_1(x) = a_1 e^{-\left(2\sqrt{\lambda\eta_1\mu\eta_2}+\mu q_2\right)x}, \tag{4-101}$$

$$\begin{aligned}p_n(x) &= a_n e^{-\left(2\sqrt{\lambda\eta_1\mu\eta_2}+\mu q_2\right)x} + \lambda\eta_1 e^{-\left(2\sqrt{\lambda\eta_1\mu\eta_2}+\mu q_2\right)x} \\ &\quad \times \int_0^x e^{\left(2\sqrt{\lambda\eta_1\mu\eta_2}+\mu q_2\right)\tau} p_{n-1}(\tau)d\tau, \ n \geq 2.\end{aligned} \tag{4-102}$$

By inserting (4-101) into (4-96) we deduce

$$\begin{aligned}
&\left(2\sqrt{\lambda\eta_1\mu\eta_2}-\mu\eta_2\right)p_0=\mu\eta_2\int_0^\infty a_1 e^{-\left(2\sqrt{\lambda\eta_1\mu\eta_2}+\mu q_2\right)x}dx\\
&=\frac{\mu\eta_2}{2\sqrt{\lambda\eta_1\mu\eta_2}+\mu q_2}a_1\\
&\Longrightarrow\\
a_1&=\frac{\left(2\sqrt{\lambda\eta_1\mu\eta_2}-\mu\eta_2\right)\left(2\sqrt{\lambda\eta_1\mu\eta_2}+\mu q_2\right)}{\mu\eta_2}p_0\\
&=\frac{4\lambda\eta_1\mu\eta_2-\mu^2\eta_2 q_2+2\sqrt{\lambda\eta_1\mu\eta_2}(\mu q_2-\mu\eta_2)}{\mu\eta_2}p_0\\
&=\left[(4\lambda\eta_1-\mu q_2)+2\sqrt{\frac{\lambda\eta_1}{\mu\eta_2}}(\mu q_2-\mu\eta_2)\right]p_0. \qquad (4\text{-}103)
\end{aligned}$$

Through combining (4-103), (4-101) and (4-102) with (4-99) and using the Fubini theorem (see Theorem 1.49) we calculate

$$\begin{aligned}
a_1&=p_1(0)\\
&=\lambda\eta_1 p_0+\mu q_2\int_0^\infty a_1 e^{-\left(2\sqrt{\lambda\eta_1\mu\eta_2}+\mu q_2\right)x}dx\\
&\quad+\mu\eta_2\int_0^\infty a_2 e^{-\left(2\sqrt{\lambda\eta_1\mu\eta_2}+\mu q_2\right)x}dx\\
&\quad+\lambda\eta_1\mu\eta_2\int_0^\infty p_1(\tau)e^{\left(2\sqrt{\lambda\eta_1\mu\eta_2}+\mu q_2\right)\tau}\\
&\qquad\times\int_\tau^\infty e^{-\left(2\sqrt{\lambda\eta_1\mu\eta_2}+\mu q_2\right)x}dxd\tau\\
&=\lambda\eta_1 p_0+\mu q_2\int_0^\infty a_1 e^{-\left(2\sqrt{\lambda\eta_1\mu\eta_2}+\mu q_2\right)x}dx\\
&\quad+\mu\eta_2\int_0^\infty a_2 e^{-\left(2\sqrt{\lambda\eta_1\mu\eta_2}+\mu q_2\right)x}dx\\
&\quad+\frac{\lambda\eta_1\mu\eta_2}{2\sqrt{\lambda\eta_1\mu\eta_2}+\mu q_2}\int_0^\infty p_1(\tau)d\tau\\
&=\lambda\eta_1 p_0+\frac{\mu q_2}{2\sqrt{\lambda\eta_1\mu\eta_2}+\mu q_2}a_1+\frac{\mu\eta_2}{2\sqrt{\lambda\eta_1\mu\eta_2}+\mu q_2}a_2\\
&\quad+\frac{\lambda\eta_1\mu\eta_2}{\left(2\sqrt{\lambda\eta_1\mu\eta_2}+\mu q_2\right)^2}a_1\\
&=\lambda\eta_1 p_0+\frac{\mu q_2\left(2\sqrt{\lambda\eta_1\mu\eta_2}+\mu q_2\right)+\lambda\eta_1\mu\eta_2}{\left(2\sqrt{\lambda\eta_1\mu\eta_2}+\mu q_2\right)^2}a_1\\
&\quad+\frac{\mu\eta_2}{2\sqrt{\lambda\eta_1\mu\eta_2}+\mu q_2}a_2
\end{aligned}$$

$$\begin{aligned}
&\Longrightarrow \\
&\frac{\mu\eta_2}{2\sqrt{\lambda\eta_1\mu\eta_2}+\mu q_2}a_2 \\
&= \left[1 - \frac{\mu q_2\left(2\sqrt{\lambda\eta_1\mu\eta_2}+\mu q_2\right)+\lambda\eta_1\mu\eta_2}{\left(2\sqrt{\lambda\eta_1\mu\eta_2}+\mu q_2\right)^2}\right]a_1 \\
&\quad - \lambda\eta_1 p_0 \\
&= \frac{\left(2\sqrt{\lambda\eta_1\mu\eta_2}+\mu q_2\right)^2 - \mu q_2\left(2\sqrt{\lambda\eta_1\mu\eta_2}+\mu q_2\right) - \lambda\eta_1\mu\eta_2}{\left(2\sqrt{\lambda\eta_1\mu\eta_2}+\mu q_2\right)^2}a_1 \\
&\quad - \lambda\eta_1 p_0 \\
&= \left[\left(2\sqrt{\lambda\eta_1\mu\eta_2}+\mu q_2\right)^2 - \mu q_2\left(2\sqrt{\lambda\eta_1\mu\eta_2}+\mu q_2\right) - \lambda\eta_1\mu\eta_2\right] \\
&\qquad \times \frac{\left(2\sqrt{\lambda\eta_1\mu\eta_2}-\mu\eta_2\right)}{\left(2\sqrt{\lambda\eta_1\mu\eta_2}+\mu q_2\right)\mu\eta_2}p_0 \\
&\quad - \lambda\eta_1 p_0 \\
&\Longrightarrow
\end{aligned}$$

$$\begin{aligned}
a_2 &= \frac{\left(3\lambda\eta_1\mu\eta_2 + 2\mu q_2\sqrt{\lambda\eta_1\mu\eta_2}\right)\left(2\sqrt{\lambda\eta_1\mu\eta_2}-\mu\eta_2\right)}{(\mu\eta_2)^2}p_0 \\
&\quad - \frac{2\sqrt{\lambda\eta_1\mu\eta_2}+\mu q_2}{\mu\eta_2}\lambda\eta_1 p_0 \\
&= \frac{1}{(\mu\eta_2)^2}p_0 \\
&\qquad \times \Big\{4\lambda\eta_1\mu\eta_2\sqrt{\lambda\eta_1\mu\eta_2} + 3\lambda\eta_1\mu\eta_2\mu q_2 \\
&\qquad -3\lambda\eta_1(\mu\eta_2)^2 - 2\mu q_2\mu\eta_2\sqrt{\lambda\eta_1\mu\eta_2}\Big\} \\
&= \frac{3\lambda\eta_1\mu q_2 - 3\lambda\eta_1\mu\eta_2 + \sqrt{\lambda\eta_1\mu\eta_2}(4\lambda\eta_1 - 2\mu q_2)}{\mu\eta_2}p_0 \\
&= \left[(4\lambda\eta_1 - 2\mu q_2) + 3\sqrt{\frac{\lambda\eta_1}{\mu\eta_2}}(\mu q_2 - \mu\eta_2)\right]\sqrt{\frac{\lambda\eta_1}{\mu\eta_2}}p_0. \qquad (4\text{-}104)
\end{aligned}$$

From (4-104), (4-101), (4-100), (4-102), (4-103) and the Fubini theorem (see Theorem 1.49) we derive

$$\begin{aligned}
a_2 &= p_2(0) \\
&= \frac{\mu q_2}{2\sqrt{\lambda\eta_1\mu\eta_2}+\mu q_2}a_2 \\
&\quad + \lambda\eta_1\mu q_2\int_0^\infty e^{-\left(2\sqrt{\lambda\eta_1\mu\eta_2}+\mu q_2\right)x}
\end{aligned}$$

$$
\begin{aligned}
&\quad \times \int_0^x p_1(\tau) e^{\left(2\sqrt{\lambda\eta_1\mu\eta_2}+\mu q_2\right)\tau} d\tau dx \\
&\quad + \frac{\mu\eta_2}{2\sqrt{\lambda\eta_1\mu\eta_2}+\mu q_2} a_3 \\
&\quad + \lambda\eta_1\mu\eta_2 \int_0^\infty e^{-\left(2\sqrt{\lambda\eta_1\mu\eta_2}+\mu q_2\right)x} \\
&\quad \times \int_0^x p_2(\tau) e^{\left(2\sqrt{\lambda\eta_1\mu\eta_2}+\mu q_2\right)\tau} d\tau dx \\
&= \frac{\mu q_2}{2\sqrt{\lambda\eta_1\mu\eta_2}+\mu q_2} a_2 \\
&\quad + \lambda\eta_1\mu q_2 \int_0^\infty p_1(\tau) e^{\left(2\sqrt{\lambda\eta_1\mu\eta_2}+\mu q_2\right)\tau} \\
&\quad \times \int_\tau^\infty e^{-\left(2\sqrt{\lambda\eta_1\mu\eta_2}+\mu q_2\right)x} dx d\tau \\
&\quad + \frac{\mu\eta_2}{2\sqrt{\lambda\eta_1\mu\eta_2}+\mu q_2} a_3 \\
&\quad + \lambda\eta_1\mu\eta_2 \int_0^\infty p_2(\tau) e^{\left(2\sqrt{\lambda\eta_1\mu\eta_2}+\mu q_2\right)\tau} \\
&\quad \times \int_\tau^\infty e^{-\left(2\sqrt{\lambda\eta_1\mu\eta_2}+\mu q_2\right)x} dx d\tau \\
&= \frac{\mu q_2}{2\sqrt{\lambda\eta_1\mu\eta_2}+\mu q_2} a_2 + \frac{\lambda\eta_1\mu q_2}{2\sqrt{\lambda\eta_1\mu\eta_2}+\mu q_2} \int_0^\infty p_1(\tau) d\tau \\
&\quad + \frac{\mu\eta_2}{2\sqrt{\lambda\eta_1\mu\eta_2}+\mu q_2} a_3 + \frac{\lambda\eta_1\mu\eta_2}{2\sqrt{\lambda\eta_1\mu\eta_2}+\mu q_2} \int_0^\infty p_2(\tau) d\tau \\
&= \frac{\mu q_2}{2\sqrt{\lambda\eta_1\mu\eta_2}+\mu q_2} a_2 + \frac{\lambda\eta_1\mu q_2}{2\sqrt{\lambda\eta_1\mu\eta_2}+\mu q_2} \int_0^\infty p_1(\tau) d\tau \\
&\quad + \frac{\mu\eta_2}{2\sqrt{\lambda\eta_1\mu\eta_2}+\mu q_2} a_3 + \frac{\lambda\eta_1\mu\eta_2}{\left(2\sqrt{\lambda\eta_1\mu\eta_2}+\mu q_2\right)^2} a_2 \\
&\quad + \frac{(\lambda\eta_1)^2\mu\eta_2}{2\sqrt{\lambda\eta_1\mu\eta_2}+\mu q_2} \\
&\quad \times \int_0^\infty e^{-\left(2\sqrt{\lambda\eta_1\mu\eta_2}+\mu q_2\right)\tau} \int_0^\tau p_1(\xi)^{\left(2\sqrt{\lambda\eta_1\mu\eta_2}+\mu q_2\right)\xi} d\xi d\tau \\
&= \frac{\mu q_2}{2\sqrt{\lambda\eta_1\mu\eta_2}+\mu q_2} a_2 + \frac{\lambda\eta_1\mu q_2}{2\sqrt{\lambda\eta_1\mu\eta_2}+\mu q_2} \int_0^\infty p_1(\tau) d\tau \\
&\quad + \frac{\mu\eta_2}{2\sqrt{\lambda\eta_1\mu\eta_2}+\mu q_2} a_3 + \frac{\lambda\eta_1\mu\eta_2}{\left(2\sqrt{\lambda\eta_1\mu\eta_2}+\mu q_2\right)^2} a_2 \\
&\quad + \frac{(\lambda\eta_1)^2\mu\eta_2}{(2\sqrt{\lambda\eta_1\mu\eta_2}+\mu q_2)^2} \int_0^\infty p_1(\xi) d\xi
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{\mu q_2 \left(2\sqrt{\lambda\eta_1\mu\eta_2} + \mu q_2\right) + \lambda\eta_1\mu\eta_2}{\left(2\sqrt{\lambda\eta_1\mu\eta_2} + \mu q_2\right)^2} a_2 \\
&\quad + \left[\frac{\lambda\eta_1\mu q_2}{\left(2\sqrt{\lambda\eta_1\mu\eta_2} + \mu q_2\right)^2} + \frac{(\lambda\eta_1)^2\mu\eta_2}{\left(2\sqrt{\lambda\eta_1\mu\eta_2} + \mu q_2\right)^3}\right] a_1 \\
&\quad + \frac{\mu\eta_2}{2\sqrt{\lambda\eta_1\mu\eta_2} + \mu q_2} a_3 \\
&\Longrightarrow
\end{aligned}
$$

$$
\begin{aligned}
&\frac{\mu\eta_2}{2\sqrt{\lambda\eta_1\mu\eta_2} + \mu q_2} a_3 \\
&= \left[1 - \frac{\mu q_2 \left(2\sqrt{\lambda\eta_1\mu\eta_2} + \mu q_2\right) + \lambda\eta_1\mu\eta_2}{\left(2\sqrt{\lambda\eta_1\mu\eta_2} + \mu q_2\right)^2}\right] a_2 \\
&\quad - \frac{\lambda\eta_1\mu q_2 \left(2\sqrt{\lambda\eta_1\mu\eta_2} + \mu q_2\right) + (\lambda\eta_1)^2\mu\eta_2}{\left(2\sqrt{\lambda\eta_1\mu\eta_2} + \mu\eta_2\right)^3} a_1 \\
&= \frac{\left(2\sqrt{\lambda\eta_1\mu\eta_2} + \mu q_2\right)^2 - \mu q_2 \left(2\sqrt{\lambda\eta_1\mu\eta_2} + \mu q_2\right) - \lambda\eta_1\mu\eta_2}{\left(2\sqrt{\lambda\eta_1\mu\eta_2} + \mu q_2\right)^2} a_2 \\
&\quad - \frac{\lambda\eta_1\mu q_2 \left(2\sqrt{\lambda\eta_1\mu\eta_2} + \mu q_2\right) + (\lambda\eta_1)^2\mu\eta_2}{\left(2\sqrt{\lambda\eta_1\mu\eta_2} + \mu\eta_2\right)^3} a_1 \\
&\Longrightarrow
\end{aligned}
$$

$$
\begin{aligned}
a_3 &= \frac{2\sqrt{\lambda\eta_1\mu\eta_2} \left(2\sqrt{\lambda\eta_1\mu\eta_2} + \mu q_2\right) - \lambda\eta_1\mu\eta_2}{\left(2\sqrt{\lambda\eta_1\mu\eta_2} + \mu q_2\right)\mu\eta_2} a_2 \\
&\quad - \frac{\lambda\eta_1\mu q_2 \left(2\sqrt{\lambda\eta_1\mu\eta_2} + \mu q_2\right) + (\lambda\eta_1)^2\mu\eta_2}{\left(2\sqrt{\lambda\eta_1\mu\eta_2} + \mu q_2\right)^2 \mu\eta_2} a_1 \\
&= \frac{3\lambda\eta_1\mu\eta_2 + 2\mu q_2\sqrt{\lambda\eta_1\mu\eta_2}}{\left(2\sqrt{\lambda\eta_1\mu\eta_2} + \mu q_2\right)\mu\eta_2} \\
&\qquad \times \frac{3\lambda\eta_1\mu q_2 - 3\lambda\eta_1\mu\eta_2 + \sqrt{\lambda\eta_1\mu\eta_2}(4\lambda\eta_1 - 2\mu q_2)}{\mu\eta_2} p_0 \\
&\quad - \frac{\lambda\eta_1\mu q_2 \left(2\sqrt{\lambda\eta_1\mu\eta_2} + \mu q_2\right) + (\lambda\eta_1)^2\mu\eta_2}{\left(2\sqrt{\lambda\eta_1\mu\eta_2} + \mu q_2\right)^2 \mu\eta_2} \\
&\qquad \times \frac{\left(2\sqrt{\lambda\eta_1\mu\eta_2} - \mu\eta_2\right)\left(2\sqrt{\lambda\eta_1\mu\eta_2} + \mu q_2\right)}{\mu\eta_2} p_0 \\
&= \frac{1}{\left(2\sqrt{\lambda\eta_1\mu\eta_2} + \mu q_2\right)(\mu\eta_2)^2} p_0 \\
&\qquad \times \Big[13\lambda^2\eta_1^2\mu^2\eta_2 q_2 - 8\lambda^2\eta_1^2\mu^2\eta_2^2 - 3\lambda\eta_1\mu^3\eta_2 q_2^2
\end{aligned}
$$

$$
\begin{aligned}
&\quad + 10\lambda^2\eta_1^2\mu\eta_2\sqrt{\lambda\eta_1\mu\eta_2} - 10\lambda\eta_1\mu^2\eta_2 q_2\sqrt{\lambda\eta_1\mu\eta_2} \\
&\quad + 4\lambda\eta_1\mu^2 q_2^2\sqrt{\lambda\eta_1\mu\eta_2}\Big] \\
&= \frac{1}{(2\sqrt{\lambda\eta_1\mu\eta_2}+\mu q_2)(\mu\eta_2)^2}p_0 \\
&\quad \times \Big[(2\sqrt{\lambda\eta_1\mu\eta_2}+\mu q_2) \\
&\quad \times (5\lambda^2\eta_1^2\mu\eta_2 + 2\lambda\eta_1\mu^2 q_2^2 - 5\lambda\eta_1\mu^2\eta_2 q_2) \\
&\quad + 8\lambda^2\eta_1^2\mu^2\eta_2 q_2 - 2\lambda\eta_1\mu^3 q_2^3 + 2\lambda\eta_1\mu^3\eta_2 q_2^2 \\
&\quad - 8\lambda^2\eta_1^2\mu^2\eta_2^2\Big] \\
&= \frac{5\lambda^2\eta_1^2\mu\eta_2 + 2\lambda\eta_1\mu^2 q_2^2 - 5\lambda\eta_1\mu^2\eta_2 q_2}{(\mu\eta_2)^2}p_0 \\
&\quad + \frac{2\lambda\eta_1\mu(q_2-\eta_2)\left(4\lambda\eta_1\mu\eta_2 - \mu^2 q_2^2\right)}{\left(2\sqrt{\lambda\eta_1\mu\eta_2}+\mu q_2\right)(\mu\eta_2)^2}p_0 \\
&= \frac{1}{(\mu\eta_2)^2}p_0 \\
&\quad \times \Big[5\lambda^2\eta_1^2\mu\eta_2 + 2\lambda\eta_1\mu^2 q_2^2 - 5\lambda\eta_1\mu^2\eta_2 q_2 \\
&\quad + 2\lambda\eta_1\mu(q_2-\eta_2)(2\sqrt{\lambda\eta_1\mu\eta_2}-\mu q_2)\Big] \\
&= \frac{5\lambda^2\eta_1^2\mu\eta_2 - 3\lambda\eta_1\mu^2\eta_2 q_2 + 4\lambda\eta_1\sqrt{\lambda\eta_1\mu\eta_2}(\mu q_2 - \mu\eta_2)}{(\mu\eta_2)^2}p_0 \\
&= \left[(5\lambda\eta_1 - 3\mu q_2) + 4\sqrt{\frac{\lambda\eta_1}{\mu\eta_2}}(\mu q_2 - \mu\eta_2)\right]\frac{\lambda\eta_1}{\mu\eta_2}p_0. \qquad (4\text{-}105)
\end{aligned}
$$

In the same way as (4-103), (4-104), (4-105), by (4-100), (4-101), (4-102) and the Fubini theorem (see Theorem 1.49) we obtain

$$
\begin{aligned}
a_n = {} & \frac{\mu q_2}{2\sqrt{\lambda\eta_1\mu\eta_2}+\mu q_2}a_n + \frac{\lambda\eta_1\mu q_2}{(2\sqrt{\lambda\eta_1\mu\eta_2}+\mu q_2)^2}a_{n-1} \\
& + \cdots \\
& + \frac{(\lambda\eta_1)^{n-1}\mu q_2}{(2\sqrt{\lambda\eta_1\mu\eta_2}+\mu q_2)^n}a_1 + \frac{\mu\eta_2}{2\sqrt{\lambda\eta_1\mu\eta_2}+\mu q_2}a_{n+1} \\
& + \frac{\lambda\eta_1\mu\eta_2}{(2\sqrt{\lambda\eta_1\mu\eta_2}+\mu q_2)^2}a_n \\
& + \cdots \\
& + \frac{(\lambda\eta_1)^n\mu\eta_2}{(2\sqrt{\lambda\eta_1\mu\eta_2}+\mu q_2)^{n+1}}a_1, \quad n \geq 2. \qquad (4\text{-}106)
\end{aligned}
$$

$$\begin{aligned}
a_{n+1} =& \frac{\mu q_2}{2\sqrt{\lambda\eta_1\mu\eta_2}+\mu q_2}a_{n+1} + \frac{\lambda\eta_1\mu q_2}{(2\sqrt{\lambda\eta_1\mu\eta_2}+\mu q_2)^2}a_n \\
&+ \frac{(\lambda\eta_1)^2\mu q_2}{(2\sqrt{\lambda\eta_1\mu\eta_2}+\mu q_2)^3}a_{n-1} \\
&+ \cdots \\
&+ \frac{(\lambda\eta_1)^n\mu q_2}{(2\sqrt{\lambda\eta_1\mu\eta_2}+\mu q_2)^{n+1}}a_1 + \frac{\mu\eta_2}{2\sqrt{\lambda\eta_1\mu\eta_2}+\mu q_2}a_{n+2} \\
&+ \frac{\lambda\eta_1\mu\eta_2}{(2\sqrt{\lambda\eta_1\mu\eta_2}+\mu q_2)^2}a_{n+1} + \frac{(\lambda\eta_1)^2\mu\eta_2}{(2\sqrt{\lambda\eta_1\mu\eta_2}+\mu q_2)^3}a_n \\
&+ \cdots \\
&+ \frac{(\lambda\eta_1)^{n+1}\mu\eta_2}{(2\sqrt{\lambda\eta_1\mu\eta_2}+\mu q_2)^{n+2}}a_1, \quad n \geq 2.
\end{aligned} \tag{4-107}$$

Through (4-107)$-\frac{\lambda\eta_1}{2\sqrt{\lambda\eta_1\mu\eta_2}+\mu q_2}\times$(4-106) it follows that

$$\begin{aligned}
& a_{n+1} - \frac{\lambda\eta_1}{2\sqrt{\lambda\eta_1\mu\eta_2}+\mu q_2}a_n \\
&= \frac{\mu q_2}{2\sqrt{\lambda\eta_1\mu\eta_2}+\mu q_2}a_{n+1} + \frac{\mu\eta_2}{2\sqrt{\lambda\eta_1\mu\eta_2}+\mu q_2}a_{n+2}, \quad n \geq 2 \\
&\Longrightarrow \\
& a_{n+2} = 2\sqrt{\frac{\lambda\eta_1}{\mu\eta_2}}a_{n+1} - \frac{\lambda\eta_1}{\mu\eta_2}a_n, \quad n \geq 2.
\end{aligned} \tag{4-108}$$

By combining (4-104) and (4-105) with (4-108) we calculate

$$\begin{aligned}
a_4 &= 2\sqrt{\frac{\lambda\eta_1}{\mu\eta_2}}a_3 - \frac{\lambda\eta_1}{\mu\eta_2}a_2 \\
&= 2\sqrt{\frac{\lambda\eta_1}{\mu\eta_2}}\left[(5\lambda\eta_1 - 3\mu q_2) + 4\sqrt{\frac{\lambda\eta_1}{\mu\eta_2}}(\mu q_2 - \mu\eta_2)\right]\frac{\lambda\eta_1}{\mu\eta_2}p_0 \\
&\quad - \frac{\lambda\eta_1}{\mu\eta_2}\left[(4\lambda\eta_1 - 2\mu q_2) + 3\sqrt{\frac{\lambda\eta_1}{\mu\eta_2}}(\mu q_2 - \mu\eta_2)\right]\sqrt{\frac{\lambda\eta_1}{\mu\eta_2}}p_0 \\
&= \left[(6\lambda\eta_1 - 4\mu q_2) + 5\sqrt{\frac{\lambda\eta_1}{\mu\eta_2}}(\mu q_2 - \mu\eta_2)\right]\left(\frac{\lambda\eta_1}{\mu\eta_2}\right)^{\frac{3}{2}} p_0.
\end{aligned} \tag{4-109}$$

(4-104), (4-105), (4-108) and (4-109) give

$$a_n = \left[(n+2)\lambda\eta_1 - n\mu q_2 + (n+1)\sqrt{\frac{\lambda\eta_1}{\mu\eta_2}}(\mu q_2 - \mu\eta_2)\right]\left(\frac{\lambda\eta_1}{\mu\eta_2}\right)^{\frac{n-1}{2}} p_0, \quad n \geq 2. \tag{4-110}$$

In the following we will check (4-110) by mathematical induction.

(4-104), (4-105) and (4-109) show that (4-110) holds for $n = 2, 3, 4$. Assume that (4-110) holds for $k \leq n+1$, that is,

$$a_k = \left[(k+2)\lambda\eta_1 - k\mu q_2 + (k+1)\sqrt{\frac{\lambda\eta_1}{\mu\eta_2}}(\mu q_2 - \mu\eta_2)\right] \times \left(\frac{\lambda\eta_1}{\mu\eta_2}\right)^{\frac{k-1}{2}} p_0, \quad \forall k \leq n+1. \tag{4-111}$$

By using (4-108) and (4-111) we get

$$\begin{aligned} a_{n+2} &= 2\sqrt{\frac{\lambda\eta_1}{\mu\eta_2}} a_{n+1} - \frac{\lambda\eta_1}{\mu\eta_2} a_n \\ &= 2\sqrt{\frac{\lambda\eta_1}{\mu\eta_2}} \Bigg[(n+3)\lambda\eta_1 - (n+1)\mu q_2 \\ &\quad + (n+2)\sqrt{\frac{\lambda\eta_1}{\mu\eta_2}}(\mu q_2 - \mu\eta_2)\Bigg] \left(\frac{\lambda\eta_1}{\mu\eta_2}\right)^{\frac{n}{2}} p_0 \\ &\quad - \frac{\lambda\eta_1}{\mu\eta_2} \Bigg[(n+2)\lambda\eta_1 - n\mu q_2 \\ &\quad + (n+1)\sqrt{\frac{\lambda\eta_1}{\mu\eta_2}}(\mu q_2 - \mu\eta_2)\Bigg] \left(\frac{\lambda\eta_1}{\mu\eta_2}\right)^{\frac{n-1}{2}} p_0 \\ &= \Bigg[(n+4)\lambda\eta_1 - (n+2)\mu q_2 \\ &\quad + (n+3)\sqrt{\frac{\lambda\eta_1}{\mu\eta_2}}(\mu q_2 - \mu\eta_2)\Bigg] \left(\frac{\lambda\eta_1}{\mu\eta_2}\right)^{\frac{n+1}{2}} p_0. \end{aligned}$$

This shows that (4-110) holds for $k = n+2$. In other words, (4-110) is true for all $n \geq 2$.

From (4-102), (4-103), (4-104), (4-105), (4-110) and the Fubini theorem (see Theorem 1.49) we estimate

$$\begin{aligned} \|p_n\|_{L^1[0,\infty)} &= \int_0^\infty |p_n(x)| dx \\ &\leq \int_0^\infty |a_n| e^{-\left(2\sqrt{\lambda\eta_1\mu\eta_2} + \mu q_2\right)x} dx \\ &\quad + \lambda\eta_1 \int_0^\infty e^{-\left(2\sqrt{\lambda\eta_1\mu\eta_2} + \mu q_2\right)x} \end{aligned}$$

$$
\begin{aligned}
&\times \int_0^x |p_{n-1}(\tau)| e^{\left(2\sqrt{\lambda\eta_1\mu\eta_2}+\mu q_2\right)\tau} d\tau dx \\
&= \frac{1}{2\sqrt{\lambda\eta_1\mu\eta_2}+\mu q_2}|a_n| \\
&\quad + \lambda\eta_1 \int_0^\infty |p_{n-1}(\tau)| e^{\left(2\sqrt{\lambda\eta_1\mu\eta_2}+\mu q_2\right)\tau} \\
&\quad \times \int_\tau^\infty e^{-\left(2\sqrt{\lambda\eta_1\mu\eta_2}+\mu q_2\right)x} dx d\tau \\
&= \frac{1}{2\sqrt{\lambda\eta_1\mu\eta_2}+\mu q_2}|a_n| \\
&\quad + \frac{\lambda\eta_1}{2\sqrt{\lambda\eta_1\mu\eta_2}+\mu q_2} \int_0^\infty |p_{n-1}(\tau)| d\tau \\
&\le \frac{1}{2\sqrt{\lambda\eta_1\mu\eta_2}+\mu q_2}|a_n| \\
&\quad + \frac{\lambda\eta_1}{2\sqrt{\lambda\eta_1\mu\eta_2}+\mu q_2} \int_0^\infty |a_{n-1}| e^{-\left(2\sqrt{\lambda\eta_1\mu\eta_2}+\mu q_2\right)\tau} d\tau \\
&\quad + \frac{(\lambda\eta_1)^2}{2\sqrt{\lambda\eta_1\mu\eta_2}+\mu q_2} \int_0^\infty e^{-\left(2\sqrt{\lambda\eta_1\mu\eta_2}+\mu q_2\right)\tau} \\
&\quad \times \int_0^\tau |p_{n-2}(\xi)| e^{\left(2\sqrt{\lambda\eta_1\mu\eta_2}+\mu q_2\right)\xi} d\xi d\tau \\
&= \frac{1}{2\sqrt{\lambda\eta_1\mu\eta_2}+\mu q_2}|a_n| + \frac{\lambda\eta_1}{\left(2\sqrt{\lambda\eta_1\mu\eta_2}+\mu q_2\right)^2}|a_{n-1}| \\
&\quad + \frac{(\lambda\eta_1)^2}{2\sqrt{\lambda\eta_1\mu\eta_2}+\mu q_2} \int_0^\infty |p_{n-2}(\xi)| e^{\left(2\sqrt{\lambda\eta_1\mu\eta_2}+\mu q_2\right)\xi} \\
&\quad \times \int_\xi^\infty e^{-\left(2\sqrt{\lambda\eta_1\mu\eta_2}+\mu q_2\right)\tau} d\tau d\xi \\
&= \frac{1}{2\sqrt{\lambda\eta_1\mu\eta_2}+\mu q_2}|a_n| + \frac{\lambda\eta_1}{\left(2\sqrt{\lambda\eta_1\mu\eta_2}+\mu q_2\right)^2}|a_{n-1}| \\
&\quad + \frac{(\lambda\eta_1)^2}{\left(2\sqrt{\lambda\eta_1\mu\eta_2}+\mu q_2\right)^2} \int_0^\infty |p_{n-2}(\xi)| d\xi \\
&\le \frac{1}{2\sqrt{\lambda\eta_1\mu\eta_2}+\mu q_2}|a_n| + \frac{\lambda\eta_1}{\left(2\sqrt{\lambda\eta_1\mu\eta_2}+\mu q_2\right)^2}|a_{n-1}| \\
&\quad + \frac{(\lambda\eta_1)^2}{\left(2\sqrt{\lambda\eta_1\mu\eta_2}+\mu q_2\right)^3}|a_{n-2}| \\
&\quad + \frac{(\lambda\eta_1)^3}{\left(2\sqrt{\lambda\eta_1\mu\eta_2}+\mu q_2\right)^2} \int_0^\infty e^{-\left(2\sqrt{\lambda\eta_1\mu\eta_2}+\mu q_2\right)\xi}
\end{aligned}
$$

$$
\begin{aligned}
&\quad \times \int_0^{\xi} |p_{n-3}(\xi)| e^{\left(2\sqrt{\lambda\eta_1\mu\eta_2}+\mu q_2\right)\tau} d\tau d\xi \\
&= \frac{1}{2\sqrt{\lambda\eta_1\mu\eta_2}+\mu q_2}|a_n| + \frac{\lambda\eta_1}{\left(2\sqrt{\lambda\eta_1\mu\eta_2}+\mu q_2\right)^2}|a_{n-1}| \\
&\quad + \frac{(\lambda\eta_1)^2}{\left(2\sqrt{\lambda\eta_1\mu\eta_2}+\mu q_2\right)^3}|a_{n-2}| \\
&\quad + \frac{(\lambda\eta_1)^3}{\left(2\sqrt{\lambda\eta_1\mu\eta_2}+\mu q_2\right)^3}\int_0^{\infty} |p_{n-3}(x)| dx \\
&\leq \cdots \\
&\leq \frac{1}{2\sqrt{\lambda\eta_1\mu\eta_2}+\mu q_2}|a_n| + \frac{\lambda\eta_1}{\left(2\sqrt{\lambda\eta_1\mu\eta_2}+\mu q_2\right)^2}|a_{n-1}| \\
&\quad + \frac{(\lambda\eta_1)^2}{\left(2\sqrt{\lambda\eta_1\mu\eta_2}+\mu q_2\right)^3}|a_{n-2}| \\
&\quad + \cdots \\
&\quad + \frac{(\lambda\eta_1)^{n-3}}{\left(2\sqrt{\lambda\eta_1\mu\eta_2}+\mu q_2\right)^{n-2}}|a_3| \\
&\quad + \frac{(\lambda\eta_1)^{n-2}}{\left(2\sqrt{\lambda\eta_1\mu\eta_2}+\mu q_2\right)^{n-1}}|a_2| \\
&\quad + \frac{(\lambda\eta_1)^{n-1}}{\left(2\sqrt{\lambda\eta_1\mu\eta_2}+\mu q_2\right)^{n}}|a_1| \\
&= \left|(n+2)\lambda\eta_1 - n\mu q_2 + (n+1)\sqrt{\frac{\lambda\eta_1}{\mu\eta_2}}(\mu q_2 - \mu\eta_2)\right| \\
&\quad \times \frac{1}{2\sqrt{\lambda\eta_1\mu\eta_2}+\mu q_2}\left(\frac{\lambda\eta_1}{\mu\eta_2}\right)^{\frac{n-1}{2}} |p_0| \\
&\quad + \left|(n+1)\lambda\eta_1 - (n-1)\mu q_2 + n\sqrt{\frac{\lambda\eta_1}{\mu\eta_2}}(\mu q_2 - \mu\eta_2)\right| \\
&\quad \times \frac{\lambda\eta_1}{\left(2\sqrt{\lambda\eta_1\mu\eta_2}+\mu q_2\right)^2}\left(\frac{\lambda\eta_1}{\mu\eta_2}\right)^{\frac{n-2}{2}} |p_0| \\
&\quad + \cdots \\
&\quad + \left|5\lambda\eta_1 - 3\mu q_2 + 4\sqrt{\frac{\lambda\eta_1}{\mu\eta_2}}(\mu q_2 - \mu\eta_2)\right| \\
&\quad \times \frac{(\lambda\eta_1)^{n-3}}{\left(2\sqrt{\lambda\eta_1\mu\eta_2}+\mu q_2\right)^{n-2}}\frac{\lambda\eta_1}{\mu\eta_2}|p_0|
\end{aligned}
$$

$$
\begin{aligned}
&+ \left| 4\lambda\eta_1 - 2\mu q_2 + 3\sqrt{\frac{\lambda\eta_1}{\mu\eta_2}}(\mu q_2 - \mu\eta_2) \right| \\
&\quad \times \frac{(\lambda\eta_1)^{n-2}}{\left(2\sqrt{\lambda\eta_1\mu\eta_2} + \mu q_2\right)^{n-1}} \left(\frac{\lambda\eta_1}{\mu\eta_2}\right)^{\frac{1}{2}} |p_0| \\
&+ \frac{(\lambda\eta_1)^{n-1}}{\left(2\sqrt{\lambda\eta_1\mu\eta_2} + \mu q_2\right)^{n}} |a_1| \\
\leq &\left[4\lambda\eta_1 \frac{(\lambda\eta_1)^{n-2}}{\left(2\sqrt{\lambda\eta_1\mu\eta_2} + \mu q_2\right)^{n-1}} \left(\frac{\lambda\eta_1}{\mu\eta_2}\right)^{\frac{1}{2}} |p_0| \right. \\
&+ 5\lambda\eta_1 \frac{(\lambda\eta_1)^{n-3}}{\left(2\sqrt{\lambda\eta_1\mu\eta_2} + \mu q_2\right)^{n-2}} \frac{\lambda\eta_1}{\mu\eta_2} |p_0| \\
&+ \cdots \\
&+ (n+1)\lambda\eta_1 \frac{\lambda\eta_1}{(2\sqrt{\lambda\eta_1\mu\eta_2} + \mu q_2)^2} \\
&\times \left(\frac{\lambda\eta_1}{\mu\eta_2}\right)^{\frac{n-2}{2}} |p_0| \\
&\left. +(n+2)\lambda\eta_1 \frac{1}{2\sqrt{\lambda\eta_1\mu\eta_2} + \mu q_2} \left(\frac{\lambda\eta_1}{\mu\eta_2}\right)^{\frac{n-1}{2}} |p_0| \right] \\
&+ \left[2\mu q_2 \frac{(\lambda\eta_1)^{n-2}}{\left(2\sqrt{\lambda\eta_1\mu\eta_2} + \mu q_2\right)^{n-1}} \left(\frac{\lambda\eta_1}{\mu\eta_2}\right)^{\frac{1}{2}} |p_0| \right. \\
&+ 3\mu q_2 \frac{(\lambda\eta_1)^{n-3}}{\left(2\sqrt{\lambda\eta_1\mu\eta_2} + \mu q_2\right)^{n-2}} \frac{\lambda\eta_1}{\mu\eta_2} |p_0| \\
&+ \cdots \\
&+ (n-1)\mu q_2 \frac{\lambda\eta_1}{(2\sqrt{\lambda\eta_1\mu\eta_2} + \mu q_2)^2} \\
&\times \left(\frac{\lambda\eta_1}{\mu\eta_2}\right)^{\frac{n-2}{2}} |p_0| \\
&\left. +n\mu q_2 \frac{1}{2\sqrt{\lambda\eta_1\mu\eta_2} + \mu q_2} \left(\frac{\lambda\eta_1}{\mu\eta_2}\right)^{\frac{n-1}{2}} |p_0| \right] \\
&+ \left[3\sqrt{\frac{\lambda\eta_1}{\mu\eta_2}} |\mu q_2 - \mu\eta_2| \frac{(\lambda\eta_1)^{n-2}}{\left(2\sqrt{\lambda\eta_1\mu\eta_2} + \mu q_2\right)^{n-1}} \right. \\
&\quad \times \left(\frac{\lambda\eta_1}{\mu\eta_2}\right)^{\frac{1}{2}} |p_0|
\end{aligned}
$$

$$
\begin{aligned}
&+4\sqrt{\frac{\lambda\eta_1}{\mu\eta_2}}|\mu q_2-\mu\eta_2|\frac{(\lambda\eta_1)^{n-3}}{\left(2\sqrt{\lambda\eta_1\mu\eta_2}+\mu q_2\right)^{n-2}}\\
&\quad\times\frac{\lambda\eta_1}{\mu\eta_2}|p_0|\\
&+\cdots\\
&+n\sqrt{\frac{\lambda\eta_1}{\mu\eta_2}}|\mu q_2-\mu\eta_2|\frac{\lambda\eta_1}{(2\sqrt{\lambda\eta_1\mu\eta_2}+\mu q_2)^2}\\
&\quad\times\left(\frac{\lambda\eta_1}{\mu\eta_2}\right)^{\frac{n-2}{2}}|p_0|\\
&+(n+1)\sqrt{\frac{\lambda\eta_1}{\mu\eta_2}}|\mu q_2-\mu\eta_2|\frac{1}{2\sqrt{\lambda\eta_1\mu\eta_2}+\mu q_2}\\
&\quad\left.\times\left(\frac{\lambda\eta_1}{\mu\eta_2}\right)^{\frac{n-1}{2}}|p_0|\right]\\
&+\frac{(\lambda\eta_1)^{n-1}}{\left(2\sqrt{\lambda\eta_1\mu\eta_2}+\mu q_2\right)^{n}}|a_1|.
\end{aligned}
\tag{4-112}
$$

For simplicity, we introduce three notations as follows.

$$
\begin{aligned}
s_1={}&4\lambda\eta_1\frac{(\lambda\eta_1)^{n-2}}{(2\sqrt{\lambda\eta_1\mu\eta_2}+\mu q_2)^{n-1}}\left(\frac{\lambda\eta_1}{\mu\eta_2}\right)^{\frac{1}{2}}|p_0|\\
&+5\lambda\eta_1\frac{(\lambda\eta_1)^{n-3}}{(2\sqrt{\lambda\eta_1\mu\eta_2}+\mu q_2)^{n-2}}\frac{\lambda\eta_1}{\mu\eta_2}|p_0|\\
&+\cdots\\
&+(n+1)\lambda\eta_1\frac{\lambda\eta_1}{(2\sqrt{\lambda\eta_1\mu\eta_2}+\mu q_2)^2}\left(\frac{\lambda\eta_1}{\mu\eta_2}\right)^{\frac{n-2}{2}}|p_0|\\
&+(n+2)\lambda\eta_1\frac{1}{2\sqrt{\lambda\eta_1\mu\eta_2}+\mu q_2}\left(\frac{\lambda\eta_1}{\mu\eta_2}\right)^{\frac{n-1}{2}}|p_0|,
\end{aligned}
\tag{4-113}
$$

$$
\begin{aligned}
s_2={}&2\mu q_2\frac{(\lambda\eta_1)^{n-2}}{(2\sqrt{\lambda\eta_1\mu\eta_2}+\mu q_2)^{n-1}}\left(\frac{\lambda\eta_1}{\mu\eta_2}\right)^{\frac{1}{2}}|p_0|\\
&+3\mu q_2\frac{(\lambda\eta_1)^{n-3}}{(2\sqrt{\lambda\eta_1\mu\eta_2}+\mu q_2)^{n-2}}\frac{\lambda\eta_1}{\mu\eta_2}|p_0|\\
&+\cdots\\
&+(n-1)\mu q_2\frac{\lambda\eta_1}{(2\sqrt{\lambda\eta_1\mu\eta_2}+\mu q_2)^2}\left(\frac{\lambda\eta_1}{\mu\eta_2}\right)^{\frac{n-2}{2}}|p_0|
\end{aligned}
$$

$$+ n\mu q_2 \frac{1}{2\sqrt{\lambda\eta_1\mu\eta_2} + \mu q_2} \left(\frac{\lambda\eta_1}{\mu\eta_2}\right)^{\frac{n-1}{2}} |p_0|, \tag{4-114}$$

$$\begin{aligned}
s_3 = {} & 3\sqrt{\frac{\lambda\eta_1}{\mu\eta_2}}|\mu q_2 - \mu\eta_2| \frac{(\lambda\eta_1)^{n-2}}{(2\sqrt{\lambda\eta_1\mu\eta_2} + \mu q_2)^{n-1}} \\
& \times \left(\frac{\lambda\eta_1}{\mu\eta_2}\right)^{\frac{1}{2}} |p_0| \\
& + 4\sqrt{\frac{\lambda\eta_1}{\mu\eta_2}}|\mu q_2 - \mu\eta_2| \frac{(\lambda\eta_1)^{n-3}}{(2\sqrt{\lambda\eta_1\mu\eta_2} + \mu q_2)^{n-2}} \frac{\lambda\eta_1}{\mu\eta_2}|p_0| \\
& + \cdots \\
& + n\sqrt{\frac{\lambda\eta_1}{\mu\eta_2}}|\mu q_2 - \mu\eta_2| \frac{\lambda\eta_1}{(2\sqrt{\lambda\eta_1\mu\eta_2} + \mu q_2)^2} \\
& \times \left(\frac{\lambda\eta_1}{\mu\eta_2}\right)^{\frac{n-2}{2}} |p_0| \\
& + (n+1)\sqrt{\frac{\lambda\eta_1}{\mu\eta_2}}|\mu q_2 - \mu\eta_2| \frac{1}{2\sqrt{\lambda\eta_1\mu\eta_2} + \mu q_2} \\
& \times \left(\frac{\lambda\eta_1}{\mu\eta_2}\right)^{\frac{n-1}{2}} |p_0|.
\end{aligned} \tag{4-115}$$

By multiplying $\frac{2\sqrt{\lambda\eta_1\mu\eta_2}+\mu q_2}{\lambda\eta_1}\left(\frac{\lambda\eta_1}{\mu\eta_2}\right)^{\frac{1}{2}}$ to two sides of (4-113) we get

$$\begin{aligned}
& \frac{2\sqrt{\lambda\eta_1\mu\eta_2} + \mu q_2}{\lambda\eta_1} \left(\frac{\lambda\eta_1}{\mu\eta_2}\right)^{\frac{1}{2}} s_1 \\
& = 4\lambda\eta_1 \frac{(\lambda\eta_1)^{n-3}}{(2\sqrt{\lambda\eta_1\mu\eta_2} + \mu q_2)^{n-2}} \frac{\lambda\eta_1}{\mu\eta_2}|p_0| \\
& \quad + 5\lambda\eta_1 \frac{(\lambda\eta_1)^{n-4}}{(2\sqrt{\lambda\eta_1\mu\eta_2} + \mu q_2)^{n-3}} \left(\frac{\lambda\eta_1}{\mu\eta_2}\right)^{\frac{3}{2}} |p_0| \\
& \quad + \cdots \\
& \quad + (n+1)\lambda\eta_1 \frac{1}{2\sqrt{\lambda\eta_1\mu\eta_2} + \mu q_2} \left(\frac{\lambda\eta_1}{\mu\eta_2}\right)^{\frac{n-1}{2}} |p_0| \\
& \quad + (n+2) \left(\frac{\lambda\eta_1}{\mu\eta_2}\right)^{\frac{n}{2}} |p_0|.
\end{aligned} \tag{4-116}$$

(4-116) minus (4-113) and noting $\lambda \geq 0$, $\mu \geq 0$, $\eta_1 \geq 0$, $\eta_2 \geq 0$, $q_2 \geq 0$ we immediately get

$$\begin{aligned}
&\frac{\sqrt{\lambda\eta_1\mu\eta_2}+\mu q_2}{\sqrt{\lambda\eta_1\mu\eta_2}} s_1 \\
&= -4\lambda\eta_1 \frac{(\lambda\eta_1)^{n-2}}{(2\sqrt{\lambda\eta_1\mu\eta_2}+\mu q_2)^{n-1}} \left(\frac{\lambda\eta_1}{\mu\eta_2}\right)^{\frac{1}{2}} |p_0| \\
&\quad - \left[\frac{(\lambda\eta_1)^{n-3}}{(2\sqrt{\lambda\eta_1\mu\eta_2}+\mu q_2)^{n-2}} \frac{\lambda\eta_1}{\mu\eta_2} \right. \\
&\quad + \cdots \\
&\quad \left. + \frac{1}{2\sqrt{\lambda\eta_1\mu\eta_2}+\mu q_2} \left(\frac{\lambda\eta_1}{\mu\eta_2}\right)^{\frac{n-1}{2}} \right] \lambda\eta_1 |p_0| \\
&\quad + (n+2) \left(\frac{\lambda\eta_1}{\mu\eta_2}\right)^{\frac{n}{2}} |p_0| \\
&\leq (n+2) \left(\frac{\lambda\eta_1}{\mu\eta_2}\right)^{\frac{n}{2}} |p_0| \\
&\Longrightarrow
\end{aligned}$$

$$s_1 \leq (n+2) \left(\frac{\lambda\eta_1}{\mu\eta_2}\right)^{\frac{n}{2}} \frac{\sqrt{\lambda\eta_1\mu\eta_2}}{\sqrt{\lambda\eta_1\mu\eta_2}+\mu q_2} |p_0| \xlongequal{\text{set}} S_n^{(1)}. \tag{4-117}$$

In the same way as (4-117), we estimate (4-114) and (4-115) as follows.

$$s_2 \leq n \left(\frac{\lambda\eta_1}{\mu\eta_2}\right)^{\frac{n}{2}} \frac{\sqrt{\lambda\eta_1\mu\eta_2}}{\sqrt{\lambda\eta_1\mu\eta_2}+\mu q_2} \frac{\mu q_2}{\lambda\eta_1} |p_0| \xlongequal{\text{set}} S_n^{(2)}, \tag{4-118}$$

$$s_3 \leq (n+1) \left(\frac{\lambda\eta_1}{\mu\eta_2}\right)^{\frac{n}{2}} \frac{|\mu q_2 - \mu\eta_2|}{\sqrt{\lambda\eta_1\mu\eta_2}+\mu q_2} |p_0| \xlongequal{\text{set}} S_n^{(3)}. \tag{4-119}$$

(4-117), (4-118) and (4-119) together with (4-112) give

$$\begin{aligned}
\|p_n\|_{L^1[0,\infty)} &\leq S_n^{(1)} + S_n^{(2)} + S_n^{(3)} \\
&\quad + \frac{(\lambda\eta_1)^{n-1}}{(2\sqrt{\lambda\eta_1\mu\eta_2}+\mu q_2)^n} |a_1| \qquad (4\text{-}120) \\
&\Longrightarrow \\
\|p\| &= |p_0| + \sum_{n=1}^{\infty} \|p_n\|_{L^1[0,\infty)} \\
&\leq |p_0| + \sum_{n=1}^{\infty} \Bigg(S_n^{(1)} + S_n^{(2)} + S_n^{(3)} \\
&\quad + \frac{(\lambda\eta_1)^{n-1}}{(2\sqrt{\lambda\eta_1\mu\eta_2}+\mu q_2)^n} |a_1| \Bigg)
\end{aligned}$$

$$
\begin{aligned}
&= |p_0| + \sum_{n=1}^{\infty} S_n^{(1)} + \sum_{n=1}^{\infty} S_n^{(2)} + \sum_{n=1}^{\infty} S_n^{(3)} \\
&\quad + |a_1| \sum_{n=1}^{\infty} \frac{(\lambda\eta_1)^{n-1}}{(2\sqrt{\lambda\eta_1\mu\eta_2} + \mu q_2)^n}.
\end{aligned}
\tag{4-121}
$$

From (4-117) we know

$$
\begin{aligned}
\sum_{n=1}^{\infty} S_n^{(1)} &= \sum_{n=1}^{\infty} \left[(n+2) \left(\frac{\lambda\eta_1}{\mu\eta_2} \right)^{\frac{n}{2}} \frac{\sqrt{\lambda\eta_1\mu\eta_2}}{\sqrt{\lambda\eta_1\mu\eta_2} + \mu q_2} |p_0| \right] \\
&= \frac{\sqrt{\lambda\eta_1\mu\eta_2}}{\sqrt{\lambda\eta_1\mu\eta_2} + \mu q_2} |p_0| \sum_{n=1}^{\infty} (n+2) \left(\frac{\lambda\eta_1}{\mu\eta_2} \right)^{\frac{n}{2}}.
\end{aligned}
\tag{4-122}
$$

In the following we will estimate (4-122). By multiplying $\left(\frac{\lambda\eta_1}{\mu\eta_2}\right)^{\frac{1}{2}}$ to two sides of

$$
\begin{aligned}
&\sum_{k=1}^{n} (k+2) \left(\frac{\lambda\eta_1}{\mu\eta_2} \right)^{\frac{k}{2}} \\
&= 3 \left(\frac{\lambda\eta_1}{\mu\eta_2} \right)^{\frac{1}{2}} + 4 \frac{\lambda\eta_1}{\mu\eta_2} + \cdots + (n+2) \left(\frac{\lambda\eta_1}{\mu\eta_2} \right)^{\frac{n}{2}}
\end{aligned}
$$

we obtain

$$
\begin{aligned}
&\left(\frac{\lambda\eta_1}{\mu\eta_2} \right)^{\frac{1}{2}} \sum_{k=1}^{n} (k+2) \left(\frac{\lambda\eta_1}{\mu\eta_2} \right)^{\frac{k}{2}} \\
&= 3 \frac{\lambda\eta_1}{\mu\eta_2} + 4 \left(\frac{\lambda\eta_1}{\mu\eta_2} \right)^{\frac{3}{2}} + \cdots + (n+2) \left(\frac{\lambda\eta_1}{\mu\eta_2} \right)^{\frac{n+1}{2}} \\
&\Longrightarrow \\
&\left(1 - \sqrt{\frac{\lambda\eta_1}{\mu\eta_2}} \right) \sum_{k=1}^{n} (k+2) \left(\frac{\lambda\eta_1}{\mu\eta_2} \right)^{\frac{k}{2}} \\
&= 3 \left(\frac{\lambda\eta_1}{\mu\eta_2} \right)^{\frac{1}{2}} + \left[\frac{\lambda\eta_1}{\mu\eta_2} + \left(\frac{\lambda\eta_1}{\mu\eta_2} \right)^{\frac{3}{2}} \right. \\
&\quad \left. + \cdots + \left(\frac{\lambda\eta_1}{\mu\eta_2} \right)^{\frac{n}{2}} \right] \\
&\quad - (n+2) \left(\frac{\lambda\eta_1}{\mu\eta_2} \right)^{\frac{n+1}{2}} \\
&= 3 \left(\frac{\lambda\eta_1}{\mu\eta_2} \right)^{\frac{1}{2}} + \frac{\frac{\lambda\eta_1}{\mu\eta_2} - \left(\frac{\lambda\eta_1}{\mu\eta_2} \right)^{\frac{n+1}{2}}}{1 - \sqrt{\frac{\lambda\eta_1}{\mu\eta_2}}} - (n+2) \left(\frac{\lambda\eta_1}{\mu\eta_2} \right)^{\frac{n+1}{2}}
\end{aligned}
$$

$$\begin{aligned}
&\Longrightarrow \\
&\sum_{k=1}^{n}(k+2)\left(\frac{\lambda\eta_1}{\mu\eta_2}\right)^{\frac{k}{2}} \\
&\le 3\frac{\sqrt{\mu\eta_2}}{\sqrt{\mu\eta_2}-\sqrt{\lambda\eta_1}}\left(\frac{\lambda\eta_1}{\mu\eta_2}\right)^{\frac{1}{2}}+\frac{\frac{\lambda\eta_1}{\mu\eta_2}-\left(\frac{\lambda\eta_1}{\mu\eta_2}\right)^{\frac{n+1}{2}}}{\left(1-\sqrt{\frac{\lambda\eta_1}{\mu\eta_2}}\right)^2} \\
&\le \frac{3\sqrt{\lambda\eta_1}}{\sqrt{\mu\eta_2}-\sqrt{\lambda\eta_1}}+\frac{\lambda\eta_1}{(\sqrt{\mu\eta_2}-\sqrt{\lambda\eta_1})^2}.
\end{aligned}$$

This shows that $\sum\limits_{k=1}^{n}(k+2)\left(\frac{\lambda\eta_1}{\mu\eta_2}\right)^{\frac{k}{2}}$ is bounded. Hence, from the convergence theorem of positive number series and (4-122) it follows that

$$\sum_{n=1}^{\infty} S_n^{(1)} < \infty. \tag{4-123}$$

By similar argument, (4-118) and (4-119) satisfy

$$\sum_{n=1}^{\infty} S_n^{(2)} < \infty, \quad \sum_{n=1}^{\infty} S_n^{(3)} < \infty. \tag{4-124}$$

It is easy to calculate

$$\begin{aligned}
&\sum_{n=1}^{\infty}\frac{(\lambda\eta_1)^{n-1}}{(2\sqrt{\lambda\eta_1\mu\eta_2}+\mu q_2)^n}|a_1| \\
&=\sum_{n=1}^{\infty}\frac{(\lambda\eta_1)^{n-1}|p_0|}{(2\sqrt{\lambda\eta_1\mu\eta_2}+\mu q_2)^n}\left|4\lambda\eta_1-\mu q_2+2\sqrt{\frac{\lambda\eta_1}{\mu q_2}}(\mu q_2-\mu\eta_2)\right| \\
&=\frac{|p_0|}{2\sqrt{\lambda\eta_1\mu\eta_2}+\mu q_2-\lambda\eta_1}\left|4\lambda\eta_1-\mu q_2+2\sqrt{\frac{\lambda\eta_1}{\mu q_2}}(\mu q_2-\mu\eta_2)\right| \\
&<\infty.
\end{aligned} \tag{4-125}$$

Summarizing (4-125), (4-124), (4-123) and (4-121) we conclude

$$\|p\| < \infty,$$

which shows that $2\sqrt{\lambda\eta_1\mu\eta_2}-\lambda\eta_1-\mu\eta_2$ is an eigenvalue of $A+U+E$ (see Definition 1.22). Moreover, from (4-101), (4-102), (4-103) and (4-110) we know that the eigenvectors corresponding to $2\sqrt{\lambda\eta_1\mu\eta_2}-\lambda\eta_1-\mu\eta_2$ span a one-dimensional linear space, that is, the geometric multiplicity of $2\sqrt{\lambda\eta_1\mu\eta_2}-\lambda\eta_1-\mu\eta_2$ is 1 (see Definition 1.23). $\square$

Lemma 4.8. *If $\lambda\eta_1 < \mu\eta_2$, then* 0 *is an eigenvalue of* $(A+U+E)^*$ *with geometric multiplicity* 1.

Proof. Consider the equation $(A+U+E)^* q^*(x) = (G+F+\mathbb{V})q^*(x) = 0$. By Lemma 4.4 it is equivalent to

$$-\lambda\eta_1 q_0^* + \lambda\eta_1 q_1^*(0) = 0, \tag{4-126}$$

$$\frac{dq_n^*(x)}{dx} - (\lambda\eta_1+\mu)q_n^*(x) + \lambda\eta_1 q_{n+1}^*(x) + \mu q_2 q_n^*(0) + \mu\eta_2 q_{n-1}^*(0) = 0, \quad n \geq 1, \tag{4-127}$$

$$q_n^*(\infty) = \alpha, \quad n \geq 1. \tag{4-128}$$

It is easy to see that

$$q^*(x) = \begin{pmatrix} \alpha \\ \alpha \\ \alpha \\ \vdots \end{pmatrix}, \quad \alpha \neq 0$$

is a solution of (4-126)–(4-128). In other words, 0 is an eigenvalue of $(A+U+E)^*$ (see Definition 1.22). Moreover, from (4-127) we deduce

$$q_2^*(x) = -\frac{1}{\lambda\eta_1}\frac{dq_1^*(x)}{dx} + \frac{\lambda\eta_1+\mu}{\lambda\eta_1}q_1^*(x) - \frac{\mu q_2}{\lambda\eta_1}q_1^*(0) - \frac{\mu\eta_2}{\lambda\eta_1}q_0^*, \tag{4-129}$$

$$q_3^*(x) = \frac{1}{(\lambda\eta_1)^2}\frac{d^2q_1^*(x)}{dx^2} - \frac{2(\lambda\eta_1+\mu)}{(\lambda\eta_1)^2}\frac{dq_1^*(x)}{dx} + \left(\frac{\lambda\eta_1+\mu}{\lambda\eta_1}\right)^2 q_1^*(x) - \frac{\mu q_2}{\lambda\eta_1}q_2^*(0) - \frac{\mu(\lambda\eta_1+\mu q_2)}{(\lambda\eta_1)^2}q_1^*(0) - \frac{\mu\eta_2(\lambda\eta_1+\mu)}{(\lambda\eta_1)^2}q_0^*, \tag{4-130}$$

$$\begin{aligned} q_4^*(x) = &-\frac{1}{(\lambda\eta_1)^4}\frac{d^4q_1^*(x)}{dx^4} - \frac{4(\lambda\eta_1+\mu)}{(\lambda\eta_1+\mu)^4}\frac{d^3q_1^*(x)}{dx^3} \\ &+ \frac{4(\lambda\eta_1+\mu)^2}{(\lambda\eta_1)^4}\frac{d^2q_1^*(x)}{dx^2} - \frac{4(\lambda\eta_1+\mu)^3}{(\lambda\eta_1)^4}\frac{dq_1^*(x)}{dx} \\ &+ \left(\frac{\lambda\eta_1+\mu}{\lambda\eta_1}\right)^4 q_1^*(x) - \frac{\mu q_2}{\lambda\eta_1}q_4^*(0) \\ &- \frac{\mu(\lambda\eta_1+\mu q_2)}{(\lambda\eta_1)^2}q_3^*(0) \end{aligned}$$

$$
\begin{aligned}
&- \frac{\mu(\lambda\eta_1+\mu)(\lambda\eta_1+\mu q_2)}{(\lambda\eta_1)^3} q_2^*(0) \\
&- \frac{\mu(\lambda\eta_1+\mu)^2(\lambda\eta_1+\mu q_2)}{(\lambda\eta_1)^4} q_1^*(0) \\
&- \frac{\mu\eta_2(\lambda\eta_1+\mu)^3}{(\lambda\eta_1)^4} q_0^*,
\end{aligned}
\tag{4-131}
$$

$\cdots$

$$
\begin{aligned}
q_n^*(x) = {} & b_1(-1)^{n-1}\frac{1}{(\lambda\eta_1)^{n-1}}\frac{d^{n-1}q^*(x)}{dx^{n-1}} \\
&- b_2\frac{\lambda\eta_1+\mu}{(\lambda\eta_1)^{n-1}}\frac{d^{n-2}q_n^*(x)}{dx^{n-1}} \\
&+ b_3\frac{(\lambda\eta_1+\mu)^2}{(\lambda\eta_1)^{n-1}}\frac{d^{n-3}q_1^*(x)}{dx^{n-3}} \\
&+ b_4\frac{(\lambda\eta_1+\mu)^3}{(\lambda\eta_1)^{n-1}}\frac{d^{n-4}q_1^*(x)}{dx^{n-4}} \\
&+ b_5\frac{(\lambda\eta_1+\mu)^4}{(\lambda\eta_1)^{n-1}}\frac{d^{n-5}q_1^*(x)}{dx^{n-5}} \\
&+ \cdots \\
&+ b_{n-1}\frac{(\lambda\eta_1+\mu)^{n-2}}{(\lambda\eta_1)^{n-1}}\frac{dq_1^*(x)}{dx} \\
&+ \left(\frac{\lambda\eta_1+\mu}{\lambda\eta_1}\right)^{n-1} q_1^*(x) - \frac{\mu q_2}{\lambda\eta_1} q_n^*(0) \\
&- \frac{\mu(\lambda\eta_1+\mu q_2)}{(\lambda\eta_1)^2} q_{n-1}^*(0) \\
&- \frac{\mu(\lambda\eta_1+\mu)(\lambda\eta_1+\mu q_2)}{(\lambda\eta_1)^3} q_{n-2}^*(0) \\
&- \frac{\mu(\lambda\eta_1+\mu)^2(\lambda\eta_1+\mu q_2)}{(\lambda\eta_1)^4} q_{n-3}^*(0) \\
&- \cdots \\
&- \frac{\mu(\lambda\eta_1+\mu)^{n-2}(\lambda\eta_1+\mu q_2)}{(\lambda\eta_1+\mu)^{n-1}} q_1^*(0) \\
&- \frac{\mu\eta_2(\lambda\eta_1+\mu)^{n-1}}{(\lambda\eta_1+\mu)^n} q_0^*, \ n \geq 2,
\end{aligned}
\tag{4-132}
$$

$\cdots$

where $b_1, b_2, \ldots, b_n$ are the same as the coefficients in the expansion of $(\xi+\zeta)^n$.

If we set for convenience $(\frac{d}{dx})^n = \frac{d^n}{dx^n}$, then (4-132) is simplified as

$$
\begin{aligned}
q_n^*(x) = & \left(\frac{\lambda\eta_1 + \mu}{\lambda\eta_1} - \frac{1}{\lambda\eta_1}\frac{d}{dx} \right)^{n-1} q_1^*(x) \\
& - \frac{\mu q_2}{\lambda\eta_1} q_n^*(0) - \frac{\mu(\lambda\eta_1 + \mu q_2)}{(\lambda\eta_1)^2} q_{n-1}^*(0) \\
& - \frac{\mu(\lambda\eta_1 + \mu)(\lambda\eta_1 + \mu q_2)}{(\lambda\eta_1)^3} q_{n-2}^*(0) \\
& - \frac{\mu(\lambda\eta_1 + \mu)^2(\lambda\eta_1 + \mu q_2)}{(\lambda\eta_1)^4} q_{n-3}^*(0) \\
& - \cdots \\
& - \frac{\mu(\lambda\eta_1 + \mu)^{n-2}(\lambda\eta_1 + \mu q_2)}{(\lambda\eta_1 + \mu)^{n-1}} q_1^*(0) \\
& - \frac{\mu\eta_2(\lambda\eta_1 + \mu)^{n-1}}{(\lambda\eta_1 + \mu)^n} q_0^*, \quad n \geq 2,
\end{aligned}
$$

from which it is not difficult to see that we can determine all $q_n^*(x)$ $(n \geq 1)$ if $q_1^*(x)$ are given. In other words, (4-126), (4-129), (4-130), (4-131) and (4-132) show that the eigenvector space corresponding to 0 is a one-dimensional linear space, that is, the geometric multiplicity of 0 is 1 (see Definition 1.23). □

Lemma 4.9. *If $\lambda\eta_1 < \mu\eta_2$, then* 0 *is an eigenvalue of $(A + U + E)^*$ with algebraic multiplicity* 1.

Proof. We use reduction to absurdity. Since 0 is an eigenvalue of $(A + U + E)^*$ by Lemma 4.8, suppose that algebraic multiplicity of 0 in X^* is larger than one. Without loss of generality, it is sufficient to prove the case that algebraic multiplicity of 0 in X^* is 2. That is, by Definition 1.23 the equation

$$(A + U + E)^* y^* = q^* \tag{4-133}$$

has a solution $y^* \in D((A + U + E)^*) = D(G)$ for the eigenvector q^* in Lemma 4.8. Without loss of generality, take it as

$$q^*(x) = \begin{pmatrix} \alpha \\ \alpha \\ \alpha \\ \vdots \end{pmatrix}, \quad \alpha \neq 0.$$

Assume that $p(x)$ is an eigenvector of $A + U + E$ corresponding to 0, that is, $p(x)$ are given by (4-93)–(4-95) (see Lemma 4.6), then by (4-133) we have

$$\langle p, (A + U + E)^* y^* \rangle = \langle p, q^* \rangle. \tag{4-134}$$

Since $(A+U+E)p=0$, then

$$0=\langle (A+U+E)p, y^*\rangle = \langle p, (A+U+E)^* y^*\rangle. \tag{4-135}$$

From (4-132) and (4-92) it may be calculated (take $p_0>0$)

$$\begin{aligned}\langle p, q^*\rangle &= \alpha p_0 + \sum_{n=1}^{\infty} \alpha p_n(x)dx\\ &= \alpha\left(p_0+\sum_{n=1}^{\infty}\int_0^{\infty} p_n(x)dx\right)\\ &= \alpha\|p\| = \alpha\frac{\mu\eta_2}{\mu\eta_2-\lambda\eta_1}p_0\\ &\neq 0.\end{aligned} \tag{4-136}$$

(4-135) and (4-136) contradict (4-134). □

By applying Lemma 4.5, Lemma 4.6, Lemma 4.9, Theorem 4.2 and Theorem 1.96 we deduce the main result in this section.

Theorem 4.10. *If $\lambda\eta_1<\mu\eta_2$, then the time-dependent solution of the system* (4-23) *converges strongly to its steady-state solution, that is,*

$$\lim_{t\to\infty}\|p(\cdot,t)-\langle q^*, p(0)\rangle p\|=0,$$

where $p(0)$ is the initial value of the system (4-23)*, p is the eigenvector of $A+U+E$ corresponding to 0, q^* is the eigenvector of $(A+U+E)^*$ corresponding to 0 and $\langle p, q^*\rangle = 1$.*

In the following we will show that the result of Theorem 4.10 is the best result on convergence of the time-dependent solution of the system (4-23).

Theorem 4.11. *If $\lambda\eta_1<\mu\eta_2$, then $(2\sqrt{\lambda\eta_1\mu\eta_2}-\lambda\eta_1-\mu\eta_2)\theta$ are eigenvalues of $(A+U+E)$ with geometric multiplicity 1 for all $\theta\in(0,1)$.*

Proof. Consider the equation

$$\left[(A+U+E)-\left(2\sqrt{\lambda\eta_1\mu\eta_2}-\lambda\eta_1-\mu\eta_2\right)\theta I\right]p=0,$$

that is,

$$\left[\lambda\eta_1+\left(2\sqrt{\lambda\eta_1\mu\eta_2}-\lambda\eta_1-\mu\eta_2\right)\theta\right]p_0=\mu\eta_2\int_0^{\infty}p_1(x)dx, \tag{4-137}$$

$$\frac{dp_1(x)}{dx}=-\left[(\lambda\eta_1+\mu)+\left(2\sqrt{\lambda\eta_1\mu\eta_2}-\lambda\eta_1-\mu\eta_2\right)\theta\right]p_1(x), \tag{4-138}$$

$$\begin{aligned}\frac{dp_n(x)}{dx}=&-\left[(\lambda\eta_1+\mu)+\left(2\sqrt{\lambda\eta_1\mu\eta_2}-\lambda\eta_1-\mu\eta_2\right)\theta\right]p_n(x)\\ &+\lambda\eta_1 p_{n-1}(x),\ n\geq 2,\end{aligned} \tag{4-139}$$

$$p_1(0) = \lambda\eta_1 p_0 + \mu q_2 \int_0^\infty p_1(x)dx + \mu\eta_2 \int_0^\infty p_2(x)dx, \tag{4-140}$$

$$p_n(0) = \mu q_2 \int_0^\infty p_n(x)dx + \mu\eta_2 \int_0^\infty p_{n+1}(x)dx, \quad n \geqslant 2. \tag{4-141}$$

By solving (4-138) and (4-139) we have

$$p_1(x) = a_1 e^{-\left[(\lambda\eta_1+\mu)+\left(2\sqrt{\lambda\eta_1\mu\eta_2}-\lambda\eta_1-\mu\eta_2\right)\theta\right]x}, \tag{4-142}$$

$$\begin{aligned} p_n(x) = {} & a_n e^{-\left[(\lambda\eta_1+\mu)+\left(2\sqrt{\lambda\eta_1\mu\eta_2}-\lambda\eta_1-\mu\eta_2\right)\theta\right]x} \\ & + \lambda\eta_1 e^{-\left[(\lambda\eta_1+\mu)+\left(2\sqrt{\lambda\eta_1\mu\eta_2}-\lambda\eta_1-\mu\eta_2\right)\theta\right]x} \\ & \times \int_0^x e^{\left[(\lambda\eta_1+\mu)+\left(2\sqrt{\lambda\eta_1\mu\eta_2}-\lambda\eta_1-\mu\eta_2\right)\theta\right]\tau} p_{n-1}(\tau)d\tau, \\ & \quad n \geq 2. \end{aligned} \tag{4-143}$$

Through inserting (4-142) into (4-137) and using

$$\begin{aligned} & \lambda\eta_1 + \mu + \left(2\sqrt{\lambda\eta_1\mu\eta_2} - \lambda\eta_1 - \mu\eta_2\right)\theta \\ & = 2\theta\sqrt{\lambda\eta_1\mu\eta_2} + (1-\theta)\lambda\eta_1 + (1-\eta_2\theta)\mu \\ & > 0 \end{aligned}$$

we calculate

$$\begin{aligned} & \left[\lambda\eta_1 + \left(2\sqrt{\lambda\eta_1\mu\eta_2} - \lambda\eta_1 - \mu\eta_2\right)\theta\right] p_0 \\ & = \mu\eta_2 \int_0^\infty a_1 e^{-\left[(\lambda\eta_1+\mu)+\left(2\sqrt{\lambda\eta_1\mu\eta_2}-\lambda\eta_1-\mu\eta_2\right)\theta\right]x} dx \\ & = \frac{\mu\eta_2}{(\lambda\eta_1+\mu) + \left(2\sqrt{\lambda\eta_1\mu\eta_2} - \lambda\eta_1 - \mu\eta_2\right)\theta} a_1 \\ & \Longrightarrow \end{aligned}$$

$$\begin{aligned} a_1 = {} & \frac{\left[\lambda\eta_1 + \left(2\sqrt{\lambda\eta_1\mu\eta_2} - \lambda\eta_1 - \mu\eta_2\right)\theta\right]}{\mu\eta_2} \\ & \times \left[(\lambda\eta_1+\mu) + \left(2\sqrt{\lambda\eta_1\mu\eta_2} - \lambda\eta_1 - \mu\eta_2\right)\theta\right] p_0. \end{aligned} \tag{4-144}$$

By inserting (4-142) and (4-143) into (4-140) and applying

$$\lambda\eta_1 + \mu + \left(2\sqrt{\lambda\eta_1\mu\eta_2} - \lambda\eta_1 - \mu\eta_2\right)\theta > 0, \quad \eta_2 + q_2 = 1,$$

(4-144) and the Fubini theorem (see Theorem 1.49) we deduce

$$\begin{aligned} a_1 & = p_1(0) \\ & = \lambda\eta_1 p_0 + \mu q_2 \int_0^\infty p_1(x)dx + \mu\eta_2 \int_0^\infty p_2(x)dx \end{aligned}$$

$$
\begin{aligned}
&= \lambda\eta_1 p_0 + \mu q_2 \int_0^\infty p_1(x)dx \\
&\quad + \mu\eta_2 \int_0^\infty a_2 e^{-\left[(\lambda\eta_1+\mu)+\left(2\sqrt{\lambda\eta_1\mu\eta_2}-\lambda\eta_1-\mu\eta_2\right)\theta\right]x}dx \\
&\quad + \lambda\eta_1\mu\eta_2 \int_0^\infty e^{-\left[(\lambda\eta_1+\mu)+\left(2\sqrt{\lambda\eta_1\mu\eta_2}-\lambda\eta_1-\mu\eta_2\right)\theta\right]x} \\
&\quad \times \int_0^x e^{\left[(\lambda\eta_1+\mu)+\left(2\sqrt{\lambda\eta_1\mu\eta_2}-\lambda\eta_1-\mu\eta_2\right)\theta\right]\tau} p_1(\tau)d\tau dx \\
&= \lambda\eta_1 p_0 + \mu q_2 \int_0^\infty a_1 e^{-\left[(\lambda\eta_1+\mu)+\left(2\sqrt{\lambda\eta_1\mu\eta_2}-\lambda\eta_1-\mu\eta_2\right)\theta\right]x}dx \\
&\quad + \mu\eta_2 \int_0^\infty a_2 e^{-\left[(\lambda\eta_1+\mu)+\left(2\sqrt{\lambda\eta_1\mu\eta_2}-\lambda\eta_1-\mu\eta_2\right)\theta\right]x}dx \\
&\quad + \lambda\eta_1\mu\eta_2 \int_0^\infty p_1(\tau) e^{\left[(\lambda\eta_1+\mu)+\left(2\sqrt{\lambda\eta_1\mu\eta_2}-\lambda\eta_1-\mu\eta_2\right)\theta\right]\tau} \\
&\quad \times \int_\tau^\infty e^{-\left[(\lambda\eta_1+\mu)+\left(2\sqrt{\lambda\eta_1\mu\eta_2}-\lambda\eta_1-\mu\eta_2\right)\theta\right]x}dx d\tau \\
&= \lambda\eta_1 p_0 + \mu q_2 \int_0^\infty a_1 e^{-\left[(\lambda\eta_1+\mu)+\left(2\sqrt{\lambda\eta_1\mu\eta_2}-\lambda\eta_1-\mu\eta_2\right)\theta\right]x}dx \\
&\quad + \mu\eta_2 \int_0^\infty a_2 e^{-\left[(\lambda\eta_1+\mu)+\left(2\sqrt{\lambda\eta_1\mu\eta_2}-\lambda\eta_1-\mu\eta_2\right)\theta\right]x}dx \\
&\quad + \frac{\lambda\eta_1\mu\eta_2}{(\lambda\eta_1+\mu)+\left(2\sqrt{\lambda\eta_1\mu\eta_2}-\lambda\eta_1-\mu\eta_2\right)\theta} \int_0^\infty p_1(\tau)d\tau \\
&= \lambda\eta_1 p_0 + \frac{\mu q_2}{(\lambda\eta_1+\mu)+\left(2\sqrt{\lambda\eta_1\mu\eta_2}-\lambda\eta_1-\mu\eta_2\right)\theta} a_1 \\
&\quad + \frac{\mu\eta_2}{(\lambda\eta_1+\mu)+\left(2\sqrt{\lambda\eta_1\mu\eta_2}-\lambda\eta_1-\mu\eta_2\right)\theta} a_2 \\
&\quad + \frac{\lambda\eta_1\mu\eta_2}{\left[(\lambda\eta_1+\mu)+\left(2\sqrt{\lambda\eta_1\mu\eta_2}-\lambda\eta_1-\mu\eta_2\right)\theta\right]^2} a_1 \\
&\Longrightarrow
\end{aligned}
$$

$$
\begin{aligned}
a_2 = \Bigg\{ &\left[(\lambda\eta_1+\mu)+\left(2\sqrt{\lambda\eta_1\mu\eta_2}-\lambda\eta_1-\mu\eta_2\right)\theta\right]^2 \\
&- \mu q_2 \left[(\lambda\eta_1+\mu)+\left(2\sqrt{\lambda\eta_1\mu\eta_2}-\lambda\eta_1-\mu\eta_2\right)\theta\right] \\
&- \lambda\eta_1\mu\eta_2 \Bigg\} \frac{\lambda\eta_1+\left(2\sqrt{\lambda\eta_1\mu\eta_2}-\lambda\eta_1-\mu\eta_2\right)\theta}{(\mu\eta_2)^2} p_0 \\
&- \frac{\lambda\eta_1+\mu+\left(2\sqrt{\lambda\eta_1\mu\eta_2}-\lambda\eta_1-\mu\eta_2\right)\theta}{\mu\eta_2} \lambda\eta_1 p_0
\end{aligned}
$$

$$
\begin{aligned}
&= \Big\{ \Big[(\lambda\eta_1 + \mu) + \Big(2\sqrt{\lambda\eta_1\mu\eta_2} - \lambda\eta_1 - \mu\eta_2 \Big) \theta \Big]^2 \\
&\quad - \mu q_2 \Big[\mu + \Big(2\sqrt{\lambda\eta_1\mu\eta_2} - \lambda\eta_1 - \mu\eta_2 \Big) \theta \Big] - \lambda\eta_1\mu \Big\} \\
&\quad \times \frac{\lambda\eta_1 + \left(2\sqrt{\lambda\eta_1\mu\eta_2} - \lambda\eta_1 - \mu\eta_2\right)\theta}{(\mu\eta_2)^2} p_0 \\
&\quad - \frac{\lambda\eta_1 + \mu + \left(2\sqrt{\lambda\eta_1\mu\eta_2} - \lambda\eta_1 - \mu\eta_2\right)\theta}{\mu\eta_2} \lambda\eta_1 p_0. \qquad (4\text{-}145)
\end{aligned}
$$

Through applying repeatedly (4-141), (4-143), the Fubini theorem (see Theorem 1.49), $\lambda\eta_1 + \mu + \left(2\sqrt{\lambda\eta_1\mu\eta_2} - \lambda\eta_1 - \mu\eta_2\right)\theta > 0$ and $\eta_2 + q_2 = 1$ we derive

$$
\begin{aligned}
a_2 &= p_2(0) \\
&= \mu q_2 \int_0^\infty p_2(x)dx + \mu\eta_2 \int_0^\infty p_3(x)dx \\
&= \mu q_2 \int_0^\infty a_2 e^{-\left[(\lambda\eta_1+\mu)+\left(2\sqrt{\lambda\eta_1\mu\eta_2}-\lambda\eta_1-\mu\eta_2\right)\theta\right]x} dx \\
&\quad + \lambda\eta_1\mu q_2 \int_0^\infty e^{-\left[(\lambda\eta_1+\mu)+\left(2\sqrt{\lambda\eta_1\mu\eta_2}-\lambda\eta_1-\mu\eta_2\right)\theta\right]x} \\
&\quad \times \int_0^x e^{\left[(\lambda\eta_1+\mu)+\left(2\sqrt{\lambda\eta_1\mu\eta_2}-\lambda\eta_1-\mu\eta_2\right)\theta\right]\tau} p_1(\tau)d\tau dx \\
&\quad + \mu\eta_2 \int_0^\infty a_3 e^{-\left[(\lambda\eta_1+\mu)+\left(2\sqrt{\lambda\eta_1\mu\eta_2}-\lambda\eta_1-\mu\eta_2\right)\theta\right]x} dx \\
&\quad + \lambda\eta_1\mu\eta_2 \int_0^\infty e^{-\left[(\lambda\eta_1+\mu)+\left(2\sqrt{\lambda\eta_1\mu\eta_2}-\lambda\eta_1-\mu\eta_2\right)\theta\right]x} \\
&\quad \times \int_0^x e^{\left[(\lambda\eta_1+\mu)+\left(2\sqrt{\lambda\eta_1\mu\eta_2}-\lambda\eta_1-\mu\eta_2\right)\theta\right]\tau} p_2(\tau)d\tau dx \\
&= \frac{\mu q_2}{\left[(\lambda\eta_1 + \mu) + \left(2\sqrt{\lambda\eta_1\mu\eta_2} - \lambda\eta_1 - \mu\eta_2\right)\theta\right]} a_2 \\
&\quad + \lambda\eta_1\mu q_2 \int_0^\infty p_1(\tau) e^{\left[(\lambda\eta_1+\mu)+\left(2\sqrt{\lambda\eta_1\mu\eta_2}-\lambda\eta_1-\mu\eta_2\right)\theta\right]\tau} \\
&\quad \times \int_\tau^\infty e^{-\left[(\lambda\eta_1+\mu)+\left(2\sqrt{\lambda\eta_1\mu\eta_2}-\lambda\eta_1-\mu\eta_2\right)\theta\right]x} dx d\tau \\
&\quad + \frac{\mu\eta_2}{\left[(\lambda\eta_1 + \mu) + \left(2\sqrt{\lambda\eta_1\mu\eta_2} - \lambda\eta_1 - \mu\eta_2\right)\theta\right]} a_3 \\
&\quad + \lambda\eta_1\mu\eta_2 \int_0^\infty p_2(\tau) e^{\left[(\lambda\eta_1+\mu)+\left(2\sqrt{\lambda\eta_1\mu\eta_2}-\lambda\eta_1-\mu\eta_2\right)\theta\right]\tau} \\
&\quad \times \int_\tau^\infty e^{-\left[(\lambda\eta_1+\mu)+\left(2\sqrt{\lambda\eta_1\mu\eta_2}-\lambda\eta_1-\mu\eta_2\right)\theta\right]x} dx d\tau
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{\mu q_2}{\left[(\lambda\eta_1+\mu)+\left(2\sqrt{\lambda\eta_1\mu\eta_2}-\lambda\eta_1-\mu\eta_2\right)\theta\right]}a_2 \\
&\quad + \frac{\lambda\eta_1\mu q_2}{\left[(\lambda\eta_1+\mu)+\left(2\sqrt{\lambda\eta_1\mu\eta_2}-\lambda\eta_1-\mu\eta_2\right)\theta\right]}\int_0^\infty p_1(\tau)d\tau \\
&\quad + \frac{\mu\eta_2}{\left[(\lambda\eta_1+\mu)+\left(2\sqrt{\lambda\eta_1\mu\eta_2}-\lambda\eta_1-\mu\eta_2\right)\theta\right]}a_3 \\
&\quad + \frac{\lambda\eta_1\mu\eta_2}{\left[(\lambda\eta_1+\mu)+\left(2\sqrt{\lambda\eta_1\mu\eta_2}-\lambda\eta_1-\mu\eta_2\right)\theta\right]}\int_0^\infty p_2(\tau)dxd\tau \\
&= \frac{\mu q_2}{\left[(\lambda\eta_1+\mu)+\left(2\sqrt{\lambda\eta_1\mu\eta_2}-\lambda\eta_1-\mu\eta_2\right)\theta\right]}a_2 \\
&\quad + \frac{\lambda\eta_1\mu q_2}{\left[(\lambda\eta_1+\mu)+\left(2\sqrt{\lambda\eta_1\mu\eta_2}-\lambda\eta_1-\mu\eta_2\right)\theta\right]}\int_0^\infty p_1(\tau)d\tau \\
&\quad + \frac{\mu\eta_2}{\left[(\lambda\eta_1+\mu)+\left(2\sqrt{\lambda\eta_1\mu\eta_2}-\lambda\eta_1-\mu\eta_2\right)\theta\right]}a_3 \\
&\quad + \frac{\lambda\eta_1\mu\eta_2}{\left[(\lambda\eta_1+\mu)+\left(2\sqrt{\lambda\eta_1\mu\eta_2}-\lambda\eta_1-\mu\eta_2\right)\theta\right]^2}a_2 \\
&\quad + \frac{(\lambda\eta_1)^2\mu\eta_2}{\left[(\lambda\eta_1+\mu)+\left(2\sqrt{\lambda\eta_1\mu\eta_2}-\lambda\eta_1-\mu\eta_2\right)\theta\right]} \\
&\quad \times \int_0^\infty e^{-\left[(\lambda\eta_1+\mu)+\left(2\sqrt{\lambda\eta_1\mu\eta_2}-\lambda\eta_1-\mu\eta_2\right)\theta\right]\tau} \\
&\quad \times \int_0^\tau p_1(\xi)e^{\left[(\lambda\eta_1+\mu)+\left(2\sqrt{\lambda\eta_1\mu\eta_2}-\lambda\eta_1-\mu\eta_2\right)\theta\right]\xi}d\xi d\tau \\
&= \frac{\mu q_2}{\left[(\lambda\eta_1+\mu)+\left(2\sqrt{\lambda\eta_1\mu\eta_2}-\lambda\eta_1-\mu\eta_2\right)\theta\right]}a_2 \\
&\quad + \frac{\lambda\eta_1\mu q_2}{\left[(\lambda\eta_1+\mu)+\left(2\sqrt{\lambda\eta_1\mu\eta_2}-\lambda\eta_1-\mu\eta_2\right)\theta\right]}\int_0^\infty p_1(\tau)d\tau \\
&\quad + \frac{\mu\eta_2}{\left[(\lambda\eta_1+\mu)+\left(2\sqrt{\lambda\eta_1\mu\eta_2}-\lambda\eta_1-\mu\eta_2\right)\theta\right]}a_3 \\
&\quad + \frac{\lambda\eta_1\mu\eta_2}{\left[(\lambda\eta_1+\mu)+\left(2\sqrt{\lambda\eta_1\mu\eta_2}-\lambda\eta_1-\mu\eta_2\right)\theta\right]^2}a_2 \\
&\quad + \frac{(\lambda\eta_1)^2\mu\eta_2}{\left[(\lambda\eta_1+\mu)+\left(2\sqrt{\lambda\eta_1\mu\eta_2}-\lambda\eta_1-\mu\eta_2\right)\theta\right]} \\
&\quad \times \int_0^\infty p_1(\xi)e^{\left[(\lambda\eta_1+\mu)+\left(2\sqrt{\lambda\eta_1\mu\eta_2}-\lambda\eta_1-\mu\eta_2\right)\theta\right]\xi} \\
&\quad \times \int_\xi^\infty e^{-\left[(\lambda\eta_1+\mu)+\left(2\sqrt{\lambda\eta_1\mu\eta_2}-\lambda\eta_1-\mu\eta_2\right)\theta\right]\tau}d\tau d\xi \\
&= \frac{\mu q_2}{\left[(\lambda\eta_1+\mu)+\left(2\sqrt{\lambda\eta_1\mu\eta_2}-\lambda\eta_1-\mu\eta_2\right)\theta\right]}a_2
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{\lambda\eta_1\mu q_2}{\left[(\lambda\eta_1+\mu)+\left(2\sqrt{\lambda\eta_1\mu\eta_2}-\lambda\eta_1-\mu\eta_2\right)\theta\right]}\int_0^\infty p_1(\tau)d\tau \\
&+\frac{\mu\eta_2}{\left[(\lambda\eta_1+\mu)+\left(2\sqrt{\lambda\eta_1\mu\eta_2}-\lambda\eta_1-\mu\eta_2\right)\theta\right]}a_3 \\
&+\frac{\lambda\eta_1\mu\eta_2}{\left[(\lambda\eta_1+\mu)+\left(2\sqrt{\lambda\eta_1\mu\eta_2}-\lambda\eta_1-\mu\eta_2\right)\theta\right]^2}a_2 \\
&+\frac{(\lambda\eta_1)^2\mu\eta_2}{\left[(\lambda\eta_1+\mu)+\left(2\sqrt{\lambda\eta_1\mu\eta_2}-\lambda\eta_1-\mu\eta_2\right)\theta\right]^2}\int_0^\infty p_1(\xi)d\xi \\
=&\frac{\mu q_2}{\left[(\lambda\eta_1+\mu)+\left(2\sqrt{\lambda\eta_1\mu\eta_2}-\lambda\eta_1-\mu\eta_2\right)\theta\right]}a_2 \\
&+\frac{\lambda\eta_1\mu q_2}{\left[(\lambda\eta_1+\mu)+\left(2\sqrt{\lambda\eta_1\mu\eta_2}-\lambda\eta_1-\mu\eta_2\right)\theta\right]^2}a_1 \\
&+\frac{\mu\eta_2}{\left[(\lambda\eta_1+\mu)+\left(2\sqrt{\lambda\eta_1\mu\eta_2}-\lambda\eta_1-\mu\eta_2\right)\theta\right]}a_3 \\
&+\frac{\lambda\eta_1\mu\eta_2}{\left[(\lambda\eta_1+\mu)+\left(2\sqrt{\lambda\eta_1\mu\eta_2}-\lambda\eta_1-\mu\eta_2\right)\theta\right]^2}a_2 \\
&+\frac{(\lambda\eta_1)^2\mu\eta_2}{\left[(\lambda\eta_1+\mu)+\left(2\sqrt{\lambda\eta_1\mu\eta_2}-\lambda\eta_1-\mu\eta_2\right)\theta\right]^3}a_1 \\
\Longrightarrow&
\end{aligned}
$$

$$
\begin{aligned}
a_3=&\left\{\lambda\eta_1+\mu+\left(2\sqrt{\lambda\eta_1\mu\eta_2}-\lambda\eta_1-\mu\eta_2\right)\theta-\mu q_2\right. \\
&\left.-\frac{\lambda\eta_1\mu\eta_2}{\left[(\lambda\eta_1+\mu)+\left(2\sqrt{\lambda\eta_1\mu\eta_2}-\lambda\eta_1-\mu\eta_2\right)\theta\right]}\right\}\frac{a_2}{\mu\eta_2} \\
&-\left\{\frac{\lambda\eta_1\mu q_2}{\left[(\lambda\eta_1+\mu)+\left(2\sqrt{\lambda\eta_1\mu\eta_2}-\lambda\eta_1-\mu\eta_2\right)\theta\right]}\right. \\
&\left.+\frac{(\lambda\eta_1)^2\mu\eta_2}{\left[(\lambda\eta_1+\mu)+\left(2\sqrt{\lambda\eta_1\mu\eta_2}-\lambda\eta_1-\mu\eta_2\right)\theta\right]^2}\right\}\frac{a_1}{\mu\eta_2} \\
=&\left\{\frac{\lambda\eta_1+\mu\eta_2+\left(2\sqrt{\lambda\eta_1\mu\eta_2}-\lambda\eta_1-\mu\eta_2\right)\theta}{\mu\eta_2}\right. \\
&\left.-\frac{\lambda\eta_1}{(\lambda\eta_1+\mu)+\left(2\sqrt{\lambda\eta_1\mu\eta_2}-\lambda\eta_1-\mu\eta_2\right)\theta}\right\} \\
&\times\left\{\left\{\left[(\lambda\eta_1+\mu)+\left(2\sqrt{\lambda\eta_1\mu\eta_2}-\lambda\eta_1-\mu\eta_2\right)\theta\right]^2\right.\right. \\
&\left.-\mu q_2\left[\mu+\left(2\sqrt{\lambda\eta_1\mu\eta_2}-\lambda\eta_1-\mu\eta_2\right)\theta\right]-\lambda\eta_1\mu\right\}
\end{aligned}
$$

$$
\begin{aligned}
&\times \frac{\lambda\eta_1 + \left(2\sqrt{\lambda\eta_1\mu\eta_2} - \lambda\eta_1 - \mu\eta_2\right)\theta}{(\mu\eta_2)^2} p_0 \\
&- \frac{\lambda\eta_1 + \mu + \left(2\sqrt{\lambda\eta_1\mu\eta_2} - \lambda\eta_1 - \mu\eta_2\right)\theta}{\mu\eta_2} \lambda\eta_1 p_0 \Bigg\} \\
&- \left\{ \lambda\eta_1\mu q_2 + \frac{(\lambda\eta_1)^2\mu\eta_2}{(\lambda\eta_1 + \mu) + \left(2\sqrt{\lambda\eta_1\mu\eta_2} - \lambda\eta_1 - \mu\eta_2\right)\theta} \right\} \\
&\times \frac{\lambda\eta_1 + \left(2\sqrt{\lambda\eta_1\mu\eta_2} - \lambda\eta_1 - \mu\eta_2\right)\theta}{(\mu\eta_2)^2} p_0. \qquad (4\text{-}146)
\end{aligned}
$$

$$
\begin{aligned}
a_3 &= p_3(0) \\
&= \mu q_2 \int_0^\infty p_3(x)dx + \mu\eta_2 \int_0^\infty p_4(x)dx \\
&= \mu q_2 \int_0^\infty a_3 e^{-\left[(\lambda\eta_1+\mu)+\left(2\sqrt{\lambda\eta_1\mu\eta_2}-\lambda\eta_1-\mu\eta_2\right)\theta\right]x} dx \\
&\quad + \lambda\eta_1\mu q_2 \int_0^\infty e^{-\left[(\lambda\eta_1+\mu)+\left(2\sqrt{\lambda\eta_1\mu\eta_2}-\lambda\eta_1-\mu\eta_2\right)\theta\right]x} \\
&\quad \times \int_0^x e^{\left[(\lambda\eta_1+\mu)+\left(2\sqrt{\lambda\eta_1\mu\eta_2}-\lambda\eta_1-\mu\eta_2\right)\theta\right]\tau} p_2(\tau) d\tau dx \\
&\quad + \mu\eta_2 \int_0^\infty a_4 e^{-\left[(\lambda\eta_1+\mu)+\left(2\sqrt{\lambda\eta_1\mu\eta_2}-\lambda\eta_1-\mu\eta_2\right)\theta\right]x} dx \\
&\quad + \lambda\eta_1\mu\eta_2 \int_0^\infty e^{-\left[(\lambda\eta_1+\mu)+\left(2\sqrt{\lambda\eta_1\mu\eta_2}-\lambda\eta_1-\mu\eta_2\right)\theta\right]x} \\
&\quad \times \int_0^x e^{\left[(\lambda\eta_1+\mu)+\left(2\sqrt{\lambda\eta_1\mu\eta_2}-\lambda\eta_1-\mu\eta_2\right)\theta\right]\tau} p_3(\tau) d\tau dx \\
&= \frac{\mu q_2}{\left[(\lambda\eta_1 + \mu) + \left(2\sqrt{\lambda\eta_1\mu\eta_2} - \lambda\eta_1 - \mu\eta_2\right)\theta\right]} a_3 \\
&\quad + \lambda\eta_1\mu q_2 \int_0^\infty p_2(\tau) e^{\left[(\lambda\eta_1+\mu)+\left(2\sqrt{\lambda\eta_1\mu\eta_2}-\lambda\eta_1-\mu\eta_2\right)\theta\right]\tau} \\
&\quad \times \int_\tau^\infty e^{-\left[(\lambda\eta_1+\mu)+\left(2\sqrt{\lambda\eta_1\mu\eta_2}-\lambda\eta_1-\mu\eta_2\right)\theta\right]x} dx d\tau \\
&\quad + \frac{\mu\eta_2}{\left[(\lambda\eta_1 + \mu) + \left(2\sqrt{\lambda\eta_1\mu\eta_2} - \lambda\eta_1 - \mu\eta_2\right)\theta\right]} a_4 \\
&\quad + \lambda\eta_1\mu\eta_2 \int_0^\infty p_3(\tau) e^{\left[(\lambda\eta_1+\mu)+\left(2\sqrt{\lambda\eta_1\mu\eta_2}-\lambda\eta_1-\mu\eta_2\right)\theta\right]\tau} \\
&\quad \times \int_\tau^\infty e^{-\left[(\lambda\eta_1+\mu)+\left(2\sqrt{\lambda\eta_1\mu\eta_2}-\lambda\eta_1-\mu\eta_2\right)\theta\right]x} dx d\tau \\
&= \frac{\mu q_2}{\left[(\lambda\eta_1 + \mu) + \left(2\sqrt{\lambda\eta_1\mu\eta_2} - \lambda\eta_1 - \mu\eta_2\right)\theta\right]} a_3
\end{aligned}
$$

$$
\begin{aligned}
&+ \frac{\lambda\eta_1\mu q_2}{\left[(\lambda\eta_1+\mu)+\left(2\sqrt{\lambda\eta_1\mu\eta_2}-\lambda\eta_1-\mu\eta_2\right)\theta\right]}\int_0^\infty p_2(\tau)d\tau \\
&+ \frac{\mu\eta_2}{\left[(\lambda\eta_1+\mu)+\left(2\sqrt{\lambda\eta_1\mu\eta_2}-\lambda\eta_1-\mu\eta_2\right)\theta\right]}a_4 \\
&+ \frac{\lambda\eta_1\mu\eta_2}{\left[(\lambda\eta_1+\mu)+\left(2\sqrt{\lambda\eta_1\mu\eta_2}-\lambda\eta_1-\mu\eta_2\right)\theta\right]}\int_0^\infty p_3(\tau)d\tau \\
=& \frac{\mu q_2}{\left[(\lambda\eta_1+\mu)+\left(2\sqrt{\lambda\eta_1\mu\eta_2}-\lambda\eta_1-\mu\eta_2\right)\theta\right]}a_3 \\
&+ \frac{\lambda\eta_1\mu q_2}{\left[(\lambda\eta_1+\mu)+\left(2\sqrt{\lambda\eta_1\mu\eta_2}-\lambda\eta_1-\mu\eta_2\right)\theta\right]} \\
&\times \int_0^\infty a_2 e^{-\left[(\lambda\eta_1+\mu)+\left(2\sqrt{\lambda\eta_1\mu\eta_2}-\lambda\eta_1-\mu\eta_2\right)\theta\right]\tau}d\tau \\
&+ \frac{(\lambda\eta_1)^2\mu q_2}{\left[(\lambda\eta_1+\mu)+\left(2\sqrt{\lambda\eta_1\mu\eta_2}-\lambda\eta_1-\mu\eta_2\right)\theta\right]} \\
&\times \int_0^\infty e^{-\left[(\lambda\eta_1+\mu)+\left(2\sqrt{\lambda\eta_1\mu\eta_2}-\lambda\eta_1-\mu\eta_2\right)\theta\right]\tau} \\
&\times \int_0^\tau e^{\left[(\lambda\eta_1+\mu)+\left(2\sqrt{\lambda\eta_1\mu\eta_2}-\lambda\eta_1-\mu\eta_2\right)\theta\right]\xi}p_1(\xi)d\xi d\tau \\
&+ \frac{\mu\eta_2}{\left[(\lambda\eta_1+\mu)+\left(2\sqrt{\lambda\eta_1\mu\eta_2}-\lambda\eta_1-\mu\eta_2\right)\theta\right]}a_4 \\
&+ \frac{\lambda\eta_1\mu\eta_2}{\left[(\lambda\eta_1+\mu)+\left(2\sqrt{\lambda\eta_1\mu\eta_2}-\lambda\eta_1-\mu\eta_2\right)\theta\right]} \\
&\times \int_0^\infty a_3 e^{-\left[(\lambda\eta_1+\mu)+\left(2\sqrt{\lambda\eta_1\mu\eta_2}-\lambda\eta_1-\mu\eta_2\right)\theta\right]\tau}d\tau \\
&+ \frac{(\lambda\eta_1)^2\mu\eta_2}{\left[(\lambda\eta_1+\mu)+\left(2\sqrt{\lambda\eta_1\mu\eta_2}-\lambda\eta_1-\mu\eta_2\right)\theta\right]} \\
&\times \int_0^\infty e^{-\left[(\lambda\eta_1+\mu)+\left(2\sqrt{\lambda\eta_1\mu\eta_2}-\lambda\eta_1-\mu\eta_2\right)\theta\right]\tau} \\
&\times \int_0^\tau e^{\left[(\lambda\eta_1+\mu)+\left(2\sqrt{\lambda\eta_1\mu\eta_2}-\lambda\eta_1-\mu\eta_2\right)\theta\right]\xi}p_2(\xi)d\xi d\tau \\
=& \frac{\mu q_2}{\left[(\lambda\eta_1+\mu)+\left(2\sqrt{\lambda\eta_1\mu\eta_2}-\lambda\eta_1-\mu\eta_2\right)\theta\right]}a_3 \\
&+ \frac{\lambda\eta_1\mu q_2}{\left[(\lambda\eta_1+\mu)+\left(2\sqrt{\lambda\eta_1\mu\eta_2}-\lambda\eta_1-\mu\eta_2\right)\theta\right]^2}a_2 \\
&+ \frac{(\lambda\eta_1)^2\mu q_2}{\left[(\lambda\eta_1+\mu)+\left(2\sqrt{\lambda\eta_1\mu\eta_2}-\lambda\eta_1-\mu\eta_2\right)\theta\right]} \\
&\times \int_0^\infty p_1(\xi)e^{\left[(\lambda\eta_1+\mu)+\left(2\sqrt{\lambda\eta_1\mu\eta_2}-\lambda\eta_1-\mu\eta_2\right)\theta\right]\xi}
\end{aligned}
$$

$$
\begin{aligned}
&\times \int_{\xi}^{\infty} e^{-\left[(\lambda\eta_1+\mu)+\left(2\sqrt{\lambda\eta_1\mu\eta_2}-\lambda\eta_1-\mu\eta_2\right)\theta\right]\tau} d\tau d\xi \\
&+ \frac{\mu\eta_2}{\left[(\lambda\eta_1+\mu)+\left(2\sqrt{\lambda\eta_1\mu\eta_2}-\lambda\eta_1-\mu\eta_2\right)\theta\right]} a_4 \\
&+ \frac{\lambda\eta_1\mu\eta_2}{\left[(\lambda\eta_1+\mu)+\left(2\sqrt{\lambda\eta_1\mu\eta_2}-\lambda\eta_1-\mu\eta_2\right)\theta\right]^2} a_3 \\
&+ \frac{(\lambda\eta_1)^2\mu\eta_2}{\left[(\lambda\eta_1+\mu)+\left(2\sqrt{\lambda\eta_1\mu\eta_2}-\lambda\eta_1-\mu\eta_2\right)\theta\right]} \\
&\times \int_{0}^{\infty} p_2(\xi) e^{\left[(\lambda\eta_1+\mu)+\left(2\sqrt{\lambda\eta_1\mu\eta_2}-\lambda\eta_1-\mu\eta_2\right)\theta\right]\xi} \\
&\times \int_{\xi}^{\infty} e^{-\left[(\lambda\eta_1+\mu)+\left(2\sqrt{\lambda\eta_1\mu\eta_2}-\lambda\eta_1-\mu\eta_2\right)\theta\right]\tau} d\tau d\xi \\
= &\frac{\mu q_2}{\left[(\lambda\eta_1+\mu)+\left(2\sqrt{\lambda\eta_1\mu\eta_2}-\lambda\eta_1-\mu\eta_2\right)\theta\right]} a_3 \\
&+ \frac{\lambda\eta_1\mu q_2}{\left[(\lambda\eta_1+\mu)+\left(2\sqrt{\lambda\eta_1\mu\eta_2}-\lambda\eta_1-\mu\eta_2\right)\theta\right]^2} a_2 \\
&+ \frac{(\lambda\eta_1)^2\mu q_2}{\left[(\lambda\eta_1+\mu)+\left(2\sqrt{\lambda\eta_1\mu\eta_2}-\lambda\eta_1-\mu\eta_2\right)\theta\right]^2} \int_0^{\infty} p_1(\xi) d\xi \\
&+ \frac{\mu\eta_2}{\left[(\lambda\eta_1+\mu)+\left(2\sqrt{\lambda\eta_1\mu\eta_2}-\lambda\eta_1-\mu\eta_2\right)\theta\right]} a_4 \\
&+ \frac{\lambda\eta_1\mu\eta_2}{\left[(\lambda\eta_1+\mu)+\left(2\sqrt{\lambda\eta_1\mu\eta_2}-\lambda\eta_1-\mu\eta_2\right)\theta\right]^2} a_3 \\
&+ \frac{(\lambda\eta_1)^2\mu\eta_2}{\left[(\lambda\eta_1+\mu)+\left(2\sqrt{\lambda\eta_1\mu\eta_2}-\lambda\eta_1-\mu\eta_2\right)\theta\right]^2} \int_0^{\infty} p_2(\xi) d\xi \\
= &\frac{\mu q_2}{\left[(\lambda\eta_1+\mu)+\left(2\sqrt{\lambda\eta_1\mu\eta_2}-\lambda\eta_1-\mu\eta_2\right)\theta\right]} a_3 \\
&+ \frac{\lambda\eta_1\mu q_2}{\left[(\lambda\eta_1+\mu)+\left(2\sqrt{\lambda\eta_1\mu\eta_2}-\lambda\eta_1-\mu\eta_2\right)\theta\right]^2} a_2 \\
&+ \frac{(\lambda\eta_1)^2\mu q_2}{\left[(\lambda\eta_1+\mu)+\left(2\sqrt{\lambda\eta_1\mu\eta_2}-\lambda\eta_1-\mu\eta_2\right)\theta\right]^3} a_1 \\
&+ \frac{\mu\eta_2}{\left[(\lambda\eta_1+\mu)+\left(2\sqrt{\lambda\eta_1\mu\eta_2}-\lambda\eta_1-\mu\eta_2\right)\theta\right]} a_4 \\
&+ \frac{\lambda\eta_1\mu\eta_2}{\left[(\lambda\eta_1+\mu)+\left(2\sqrt{\lambda\eta_1\mu\eta_2}-\lambda\eta_1-\mu\eta_2\right)\theta\right]^2} a_3 \\
&+ \frac{(\lambda\eta_1)^2\mu\eta_2}{\left[(\lambda\eta_1+\mu)+\left(2\sqrt{\lambda\eta_1\mu\eta_2}-\lambda\eta_1-\mu\eta_2\right)\theta\right]^2}
\end{aligned}
$$

$$\times \int_0^\infty a_2 e^{-\left[(\lambda\eta_1+\mu)+\left(2\sqrt{\lambda\eta_1\mu\eta_2}-\lambda\eta_1-\mu\eta_2\right)\theta\right]\xi} d\xi$$
$$+ \frac{(\lambda\eta_1)^3\mu\eta_2}{\left[(\lambda\eta_1+\mu)+\left(2\sqrt{\lambda\eta_1\mu\eta_2}-\lambda\eta_1-\mu\eta_2\right)\theta\right]^2}$$
$$\times \int_0^\infty e^{-\left[(\lambda\eta_1+\mu)+\left(2\sqrt{\lambda\eta_1\mu\eta_2}-\lambda\eta_1-\mu\eta_2\right)\theta\right]\xi}$$
$$\times \int_0^\xi e^{\left[(\lambda\eta_1+\mu)+\left(2\sqrt{\lambda\eta_1\mu\eta_2}-\lambda\eta_1-\mu\eta_2\right)\theta\right]\tau} p_1(\tau) d\tau d\xi$$

$$= \frac{\mu q_2}{\left[(\lambda\eta_1+\mu)+\left(2\sqrt{\lambda\eta_1\mu\eta_2}-\lambda\eta_1-\mu\eta_2\right)\theta\right]} a_3$$
$$+ \frac{\lambda\eta_1\mu q_2}{\left[(\lambda\eta_1+\mu)+\left(2\sqrt{\lambda\eta_1\mu\eta_2}-\lambda\eta_1-\mu\eta_2\right)\theta\right]^2} a_2$$
$$+ \frac{(\lambda\eta_1)^2\mu q_2}{\left[(\lambda\eta_1+\mu)+\left(2\sqrt{\lambda\eta_1\mu\eta_2}-\lambda\eta_1-\mu\eta_2\right)\theta\right]^3} a_1$$
$$+ \frac{\mu\eta_2}{\left[(\lambda\eta_1+\mu)+\left(2\sqrt{\lambda\eta_1\mu\eta_2}-\lambda\eta_1-\mu\eta_2\right)\theta\right]} a_4$$
$$+ \frac{\lambda\eta_1\mu\eta_2}{\left[(\lambda\eta_1+\mu)+\left(2\sqrt{\lambda\eta_1\mu\eta_2}-\lambda\eta_1-\mu\eta_2\right)\theta\right]^2} a_3$$
$$+ \frac{(\lambda\eta_1)^2\mu\eta_2}{\left[(\lambda\eta_1+\mu)+\left(2\sqrt{\lambda\eta_1\mu\eta_2}-\lambda\eta_1-\mu\eta_2\right)\theta\right]^3} a_2$$
$$+ \frac{(\lambda\eta_1)^3\mu\eta_2}{\left[(\lambda\eta_1+\mu)+\left(2\sqrt{\lambda\eta_1\mu\eta_2}-\lambda\eta_1-\mu\eta_2\right)\theta\right]^2}$$
$$\times \int_0^\infty e^{\left[(\lambda\eta_1+\mu)+\left(2\sqrt{\lambda\eta_1\mu\eta_2}-\lambda\eta_1-\mu\eta_2\right)\theta\right]\tau} p_1(\tau)$$
$$\times \int_\tau^\infty e^{-\left[(\lambda\eta_1+\mu)+\left(2\sqrt{\lambda\eta_1\mu\eta_2}-\lambda\eta_1-\mu\eta_2\right)\theta\right]\xi} d\xi d\tau$$

$$= \frac{\mu q_2}{\left[(\lambda\eta_1+\mu)+\left(2\sqrt{\lambda\eta_1\mu\eta_2}-\lambda\eta_1-\mu\eta_2\right)\theta\right]} a_3$$
$$+ \frac{\lambda\eta_1\mu q_2}{\left[(\lambda\eta_1+\mu)+\left(2\sqrt{\lambda\eta_1\mu\eta_2}-\lambda\eta_1-\mu\eta_2\right)\theta\right]^2} a_2$$
$$+ \frac{(\lambda\eta_1)^2\mu q_2}{\left[(\lambda\eta_1+\mu)+\left(2\sqrt{\lambda\eta_1\mu\eta_2}-\lambda\eta_1-\mu\eta_2\right)\theta\right]^3} a_1$$
$$+ \frac{\mu\eta_2}{\left[(\lambda\eta_1+\mu)+\left(2\sqrt{\lambda\eta_1\mu\eta_2}-\lambda\eta_1-\mu\eta_2\right)\theta\right]} a_4$$
$$+ \frac{\lambda\eta_1\mu\eta_2}{\left[(\lambda\eta_1+\mu)+\left(2\sqrt{\lambda\eta_1\mu\eta_2}-\lambda\eta_1-\mu\eta_2\right)\theta\right]^2} a_3$$

$$
\begin{aligned}
&+ \frac{(\lambda\eta_1)^2\mu\eta_2}{\left[(\lambda\eta_1+\mu)+\left(2\sqrt{\lambda\eta_1\mu\eta_2}-\lambda\eta_1-\mu\eta_2\right)\theta\right]^3}a_2 \\
&+ \frac{(\lambda\eta_1)^3\mu\eta_2}{\left[(\lambda\eta_1+\mu)+\left(2\sqrt{\lambda\eta_1\mu\eta_2}-\lambda\eta_1-\mu\eta_2\right)\theta\right]^3}\int_0^\infty p_1(x)dx \\
= &\frac{\mu q_2}{\left[(\lambda\eta_1+\mu)+\left(2\sqrt{\lambda\eta_1\mu\eta_2}-\lambda\eta_1-\mu\eta_2\right)\theta\right]}a_3 \\
&+ \frac{\lambda\eta_1\mu q_2}{\left[(\lambda\eta_1+\mu)+\left(2\sqrt{\lambda\eta_1\mu\eta_2}-\lambda\eta_1-\mu\eta_2\right)\theta\right]^2}a_2 \\
&+ \frac{(\lambda\eta_1)^2\mu q_2}{\left[(\lambda\eta_1+\mu)+\left(2\sqrt{\lambda\eta_1\mu\eta_2}-\lambda\eta_1-\mu\eta_2\right)\theta\right]^3}a_1 \\
&+ \frac{\mu\eta_2}{\left[(\lambda\eta_1+\mu)+\left(2\sqrt{\lambda\eta_1\mu\eta_2}-\lambda\eta_1-\mu\eta_2\right)\theta\right]}a_4 \\
&+ \frac{\lambda\eta_1\mu\eta_2}{\left[(\lambda\eta_1+\mu)+\left(2\sqrt{\lambda\eta_1\mu\eta_2}-\lambda\eta_1-\mu\eta_2\right)\theta\right]^2}a_3 \\
&+ \frac{(\lambda\eta_1)^2\mu\eta_2}{\left[(\lambda\eta_1+\mu)+\left(2\sqrt{\lambda\eta_1\mu\eta_2}-\lambda\eta_1-\mu\eta_2\right)\theta\right]^3}a_2 \\
&+ \frac{(\lambda\eta_1)^3\mu\eta_2}{\left[(\lambda\eta_1+\mu)+\left(2\sqrt{\lambda\eta_1\mu\eta_2}-\lambda\eta_1-\mu\eta_2\right)\theta\right]^4}a_1.
\end{aligned} \tag{4-147}
$$

In the same way as (4-144), (4-145), (4-146) and (4-147), by using (4-141), (4-142), (4-143), the Fubini theorem (see Theorem 1.49) and

$$\lambda\eta_1+\mu+\left(2\sqrt{\lambda\eta_1\mu\eta_2}-\lambda\eta_1-\mu\eta_2\right)\theta>0$$

we obtain

$$
\begin{aligned}
a_n &= p_n(0) \\
&= \mu q_2\int_0^\infty p_n(x)dx+\mu\eta_2\int_0^\infty p_{n+1}(x)dx \\
&= \mu q_2\int_0^\infty a_n e^{-\left[(\lambda\eta_1+\mu)+\left(2\sqrt{\lambda\eta_1\mu\eta_2}-\lambda\eta_1-\mu\eta_2\right)\theta\right]x}dx \\
&\quad + \lambda\eta_1\mu q_2\int_0^\infty e^{-\left[(\lambda\eta_1+\mu)+\left(2\sqrt{\lambda\eta_1\mu\eta_2}-\lambda\eta_1-\mu\eta_2\right)\theta\right]x} \\
&\quad \times\int_0^x e^{\left[(\lambda\eta_1+\mu)+\left(2\sqrt{\lambda\eta_1\mu\eta_2}-\lambda\eta_1-\mu\eta_2\right)\theta\right]\tau}p_{n-1}(\tau)d\tau dx \\
&\quad + \mu\eta_2\int_0^\infty a_{n+1}e^{-\left[(\lambda\eta_1+\mu)+\left(2\sqrt{\lambda\eta_1\mu\eta_2}-\lambda\eta_1-\mu\eta_2\right)\theta\right]x}dx \\
&\quad + \lambda\eta_1\mu\eta_2\int_0^\infty e^{-\left[(\lambda\eta_1+\mu)+\left(2\sqrt{\lambda\eta_1\mu\eta_2}-\lambda\eta_1-\mu\eta_2\right)\theta\right]x}
\end{aligned}
$$

$$
\begin{aligned}
&\times \int_0^x e^{\left[(\lambda\eta_1+\mu)+\left(2\sqrt{\lambda\eta_1\mu\eta_2}-\lambda\eta_1-\mu\eta_2\right)\theta\right]\tau} p_n(\tau)d\tau dx \\
= &\frac{\mu q_2}{\left[(\lambda\eta_1+\mu)+\left(2\sqrt{\lambda\eta_1\mu\eta_2}-\lambda\eta_1-\mu\eta_2\right)\theta\right]} a_n \\
&+ \lambda\eta_1\mu q_2 \int_0^\infty p_{n-1}(\tau) e^{\left[(\lambda\eta_1+\mu)+\left(2\sqrt{\lambda\eta_1\mu\eta_2}-\lambda\eta_1-\mu\eta_2\right)\theta\right]\tau} \\
&\times \int_\tau^\infty e^{-\left[(\lambda\eta_1+\mu)+\left(2\sqrt{\lambda\eta_1\mu\eta_2}-\lambda\eta_1-\mu\eta_2\right)\theta\right]x} dx d\tau \\
&+ \frac{\mu\eta_2}{\left[(\lambda\eta_1+\mu)+\left(2\sqrt{\lambda\eta_1\mu\eta_2}-\lambda\eta_1-\mu\eta_2\right)\theta\right]} a_{n+1} \\
&+ \lambda\eta_1\mu\eta_2 \int_0^\infty p_n(\tau) e^{\left[(\lambda\eta_1+\mu)+\left(2\sqrt{\lambda\eta_1\mu\eta_2}-\lambda\eta_1-\mu\eta_2\right)\theta\right]\tau} \\
&\times \int_\tau^\infty e^{-\left[(\lambda\eta_1+\mu)+\left(2\sqrt{\lambda\eta_1\mu\eta_2}-\lambda\eta_1-\mu\eta_2\right)\theta\right]x} dx d\tau \\
= &\frac{\mu q_2}{\left[(\lambda\eta_1+\mu)+\left(2\sqrt{\lambda\eta_1\mu\eta_2}-\lambda\eta_1-\mu\eta_2\right)\theta\right]} a_n \\
&+ \frac{\lambda\eta_1\mu q_2}{\left[(\lambda\eta_1+\mu)+\left(2\sqrt{\lambda\eta_1\mu\eta_2}-\lambda\eta_1-\mu\eta_2\right)\theta\right]} \int_0^\infty p_{n-1}(\tau)d\tau \\
&+ \frac{\mu\eta_2}{\left[(\lambda\eta_1+\mu)+\left(2\sqrt{\lambda\eta_1\mu\eta_2}-\lambda\eta_1-\mu\eta_2\right)\theta\right]} a_{n+1} \\
&+ \frac{\lambda\eta_1\mu\eta_2}{\left[(\lambda\eta_1+\mu)+\left(2\sqrt{\lambda\eta_1\mu\eta_2}-\lambda\eta_1-\mu\eta_2\right)\theta\right]} \int_0^\infty p_n(\tau)d\tau \\
= &\frac{\mu q_2}{\left[(\lambda\eta_1+\mu)+\left(2\sqrt{\lambda\eta_1\mu\eta_2}-\lambda\eta_1-\mu\eta_2\right)\theta\right]} a_n \\
&+ \frac{\lambda\eta_1\mu q_2}{\left[(\lambda\eta_1+\mu)+\left(2\sqrt{\lambda\eta_1\mu\eta_2}-\lambda\eta_1-\mu\eta_2\right)\theta\right]} \\
&\times \int_0^\infty a_{n-1} e^{-\left[(\lambda\eta_1+\mu)+\left(2\sqrt{\lambda\eta_1\mu\eta_2}-\lambda\eta_1-\mu\eta_2\right)\theta\right]\tau} d\tau \\
&+ \frac{(\lambda\eta_1)^2\mu q_2}{\left[(\lambda\eta_1+\mu)+\left(2\sqrt{\lambda\eta_1\mu\eta_2}-\lambda\eta_1-\mu\eta_2\right)\theta\right]} \\
&\times \int_0^\infty e^{-\left[(\lambda\eta_1+\mu)+\left(2\sqrt{\lambda\eta_1\mu\eta_2}-\lambda\eta_1-\mu\eta_2\right)\theta\right]\tau} \\
&\times \int_0^\tau e^{\left[(\lambda\eta_1+\mu)+\left(2\sqrt{\lambda\eta_1\mu\eta_2}-\lambda\eta_1-\mu\eta_2\right)\theta\right]\xi} p_{n-2}(\xi)d\xi d\tau \\
&+ \frac{\mu\eta_2}{\left[(\lambda\eta_1+\mu)+\left(2\sqrt{\lambda\eta_1\mu\eta_2}-\lambda\eta_1-\mu\eta_2\right)\theta\right]} a_{n+1} \\
&+ \frac{\lambda\eta_1\mu\eta_2}{\left[(\lambda\eta_1+\mu)+\left(2\sqrt{\lambda\eta_1\mu\eta_2}-\lambda\eta_1-\mu\eta_2\right)\theta\right]}
\end{aligned}
$$

$$
\begin{aligned}
&\times \int_0^\infty a_n e^{-\left[(\lambda\eta_1+\mu)+\left(2\sqrt{\lambda\eta_1\mu\eta_2}-\lambda\eta_1-\mu\eta_2\right)\theta\right]\tau} d\tau \\
&+ \frac{(\lambda\eta_1)^2\mu\eta_2}{\left[(\lambda\eta_1+\mu)+\left(2\sqrt{\lambda\eta_1\mu\eta_2}-\lambda\eta_1-\mu\eta_2\right)\theta\right]} \\
&\times \int_0^\infty e^{-\left[(\lambda\eta_1+\mu)+\left(2\sqrt{\lambda\eta_1\mu\eta_2}-\lambda\eta_1-\mu\eta_2\right)\theta\right]\tau} \\
&\times \int_0^\tau e^{\left[(\lambda\eta_1+\mu)+\left(2\sqrt{\lambda\eta_1\mu\eta_2}-\lambda\eta_1-\mu\eta_2\right)\theta\right]\xi} p_{n-1}(\xi) d\xi d\tau \\
= &\frac{\mu q_2}{\left[(\lambda\eta_1+\mu)+\left(2\sqrt{\lambda\eta_1\mu\eta_2}-\lambda\eta_1-\mu\eta_2\right)\theta\right]} a_n \\
&+ \frac{\lambda\eta_1\mu q_2}{\left[(\lambda\eta_1+\mu)+\left(2\sqrt{\lambda\eta_1\mu\eta_2}-\lambda\eta_1-\mu\eta_2\right)\theta\right]^2} a_{n-1} \\
&+ \frac{(\lambda\eta_1)^2\mu q_2}{\left[(\lambda\eta_1+\mu)+\left(2\sqrt{\lambda\eta_1\mu\eta_2}-\lambda\eta_1-\mu\eta_2\right)\theta\right]} \\
&\times \int_0^\infty e^{\left[(\lambda\eta_1+\mu)+\left(2\sqrt{\lambda\eta_1\mu\eta_2}-\lambda\eta_1-\mu\eta_2\right)\theta\right]\xi} p_{n-2}(\xi) \\
&\times \int_\xi^\infty e^{-\left[(\lambda\eta_1+\mu)+\left(2\sqrt{\lambda\eta_1\mu\eta_2}-\lambda\eta_1-\mu\eta_2\right)\theta\right]\tau} d\tau d\xi \\
&+ \frac{\mu\eta_2}{\left[(\lambda\eta_1+\mu)+\left(2\sqrt{\lambda\eta_1\mu\eta_2}-\lambda\eta_1-\mu\eta_2\right)\theta\right]} a_{n+1} \\
&+ \frac{\lambda\eta_1\mu\eta_2}{\left[(\lambda\eta_1+\mu)+\left(2\sqrt{\lambda\eta_1\mu\eta_2}-\lambda\eta_1-\mu\eta_2\right)\theta\right]^2} a_n \\
&+ \frac{(\lambda\eta_1)^2\mu\eta_2}{\left[(\lambda\eta_1+\mu)+\left(2\sqrt{\lambda\eta_1\mu\eta_2}-\lambda\eta_1-\mu\eta_2\right)\theta\right]} \\
&\times \int_0^\infty e^{\left[(\lambda\eta_1+\mu)+\left(2\sqrt{\lambda\eta_1\mu\eta_2}-\lambda\eta_1-\mu\eta_2\right)\theta\right]\xi} p_{n-1}(\xi) \\
&\times \int_\xi^\infty e^{-\left[(\lambda\eta_1+\mu)+\left(2\sqrt{\lambda\eta_1\mu\eta_2}-\lambda\eta_1-\mu\eta_2\right)\theta\right]\tau} d\tau d\xi \\
= &\frac{\mu q_2}{\left[(\lambda\eta_1+\mu)+\left(2\sqrt{\lambda\eta_1\mu\eta_2}-\lambda\eta_1-\mu\eta_2\right)\theta\right]} a_n \\
&+ \frac{\lambda\eta_1\mu q_2}{\left[(\lambda\eta_1+\mu)+\left(2\sqrt{\lambda\eta_1\mu\eta_2}-\lambda\eta_1-\mu\eta_2\right)\theta\right]^2} a_{n-1} \\
&+ \frac{(\lambda\eta_1)^2\mu q_2}{\left[(\lambda\eta_1+\mu)+\left(2\sqrt{\lambda\eta_1\mu\eta_2}-\lambda\eta_1-\mu\eta_2\right)\theta\right]^2} \int_0^\infty p_{n-2}(\xi) d\xi \\
&+ \frac{\mu\eta_2}{\left[(\lambda\eta_1+\mu)+\left(2\sqrt{\lambda\eta_1\mu\eta_2}-\lambda\eta_1-\mu\eta_2\right)\theta\right]} a_{n+1}
\end{aligned}
$$

$$
\begin{aligned}
&+ \frac{\lambda\eta_1\mu\eta_2}{\left[(\lambda\eta_1+\mu)+\left(2\sqrt{\lambda\eta_1\mu\eta_2}-\lambda\eta_1-\mu\eta_2\right)\theta\right]^2} a_n \\
&+ \frac{(\lambda\eta_1)^2\mu\eta_2}{\left[(\lambda\eta_1+\mu)+\left(2\sqrt{\lambda\eta_1\mu\eta_2}-\lambda\eta_1-\mu\eta_2\right)\theta\right]^2} \int_0^\infty p_{n-1}(\xi)d\xi \\
= &\frac{\mu q_2}{\left[(\lambda\eta_1+\mu)+\left(2\sqrt{\lambda\eta_1\mu\eta_2}-\lambda\eta_1-\mu\eta_2\right)\theta\right]} a_n \\
&+ \frac{\lambda\eta_1\mu q_2}{\left[(\lambda\eta_1+\mu)+\left(2\sqrt{\lambda\eta_1\mu\eta_2}-\lambda\eta_1-\mu\eta_2\right)\theta\right]^2} a_{n-1} \\
&+ \frac{(\lambda\eta_1)^2\mu q_2}{\left[(\lambda\eta_1+\mu)+\left(2\sqrt{\lambda\eta_1\mu\eta_2}-\lambda\eta_1-\mu\eta_2\right)\theta\right]^2} \\
&\times \int_0^\infty a_{n-2} e^{-\left[(\lambda\eta_1+\mu)+\left(2\sqrt{\lambda\eta_1\mu\eta_2}-\lambda\eta_1-\mu\eta_2\right)\theta\right]x} dx \\
&+ \frac{(\lambda\eta_1)^3\mu q_2}{\left[(\lambda\eta_1+\mu)+\left(2\sqrt{\lambda\eta_1\mu\eta_2}-\lambda\eta_1-\mu\eta_2\right)\theta\right]^2} \\
&\times \int_0^\infty e^{-\left[(\lambda\eta_1+\mu)+\left(2\sqrt{\lambda\eta_1\mu\eta_2}-\lambda\eta_1-\mu\eta_2\right)\theta\right]x} \\
&\times \int_0^x p_{n-3}(\tau) e^{\left[(\lambda\eta_1+\mu)+\left(2\sqrt{\lambda\eta_1\mu\eta_2}-\lambda\eta_1-\mu\eta_2\right)\theta\right]\tau} d\tau dx \\
&+ \frac{\mu\eta_2}{\left[(\lambda\eta_1+\mu)+\left(2\sqrt{\lambda\eta_1\mu\eta_2}-\lambda\eta_1-\mu\eta_2\right)\theta\right]} a_{n+1} \\
&+ \frac{\lambda\eta_1\mu\eta_2}{\left[(\lambda\eta_1+\mu)+\left(2\sqrt{\lambda\eta_1\mu\eta_2}-\lambda\eta_1-\mu\eta_2\right)\theta\right]^2} a_n \\
&+ \frac{(\lambda\eta_1)^2\mu\eta_2}{\left[(\lambda\eta_1+\mu)+\left(2\sqrt{\lambda\eta_1\mu\eta_2}-\lambda\eta_1-\mu\eta_2\right)\theta\right]^2} \\
&\times \int_0^\infty a_{n-1} e^{-\left[(\lambda\eta_1+\mu)+\left(2\sqrt{\lambda\eta_1\mu\eta_2}-\lambda\eta_1-\mu\eta_2\right)\theta\right]x} dx \\
&+ \frac{(\lambda\eta_1)^3\mu\eta_2}{\left[(\lambda\eta_1+\mu)+\left(2\sqrt{\lambda\eta_1\mu\eta_2}-\lambda\eta_1-\mu\eta_2\right)\theta\right]^2} \\
&\times \int_0^\infty e^{-\left[(\lambda\eta_1+\mu)+\left(2\sqrt{\lambda\eta_1\mu\eta_2}-\lambda\eta_1-\mu\eta_2\right)\theta\right]x} \\
&\times \int_0^x p_{n-2}(\tau) e^{\left[(\lambda\eta_1+\mu)+\left(2\sqrt{\lambda\eta_1\mu\eta_2}-\lambda\eta_1-\mu\eta_2\right)\theta\right]\tau} d\tau dx \\
= &\frac{\mu q_2}{\left[(\lambda\eta_1+\mu)+\left(2\sqrt{\lambda\eta_1\mu\eta_2}-\lambda\eta_1-\mu\eta_2\right)\theta\right]} a_n \\
&+ \frac{\lambda\eta_1\mu q_2}{\left[(\lambda\eta_1+\mu)+\left(2\sqrt{\lambda\eta_1\mu\eta_2}-\lambda\eta_1-\mu\eta_2\right)\theta\right]^2} a_{n-1}
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{(\lambda\eta_1)^2\mu q_2}{\left[(\lambda\eta_1+\mu)+\left(2\sqrt{\lambda\eta_1\mu\eta_2}-\lambda\eta_1-\mu\eta_2\right)\theta\right]^3}a_{n-2}\\
&+\frac{(\lambda\eta_1)^3\mu q_2}{\left[(\lambda\eta_1+\mu)+\left(2\sqrt{\lambda\eta_1\mu\eta_2}-\lambda\eta_1-\mu\eta_2\right)\theta\right]^3}\int_0^\infty p_{n-3}(x)dx\\
&+\frac{\mu\eta_2}{\left[(\lambda\eta_1+\mu)+\left(2\sqrt{\lambda\eta_1\mu\eta_2}-\lambda\eta_1-\mu\eta_2\right)\theta\right]}a_{n+1}\\
&+\frac{\lambda\eta_1\mu\eta_2}{\left[(\lambda\eta_1+\mu)+\left(2\sqrt{\lambda\eta_1\mu\eta_2}-\lambda\eta_1-\mu\eta_2\right)\theta\right]^2}a_{n}\\
&+\frac{(\lambda\eta_1)^2\mu\eta_2}{\left[(\lambda\eta_1+\mu)+\left(2\sqrt{\lambda\eta_1\mu\eta_2}-\lambda\eta_1-\mu\eta_2\right)\theta\right]^3}a_{n-1}\\
&+\frac{(\lambda\eta_1)^3\mu\eta_2}{\left[(\lambda\eta_1+\mu)+\left(2\sqrt{\lambda\eta_1\mu\eta_2}-\lambda\eta_1-\mu\eta_2\right)\theta\right]^3}\int_0^\infty p_{n-2}(x)dx\\
=&\cdots\\
=&\frac{\mu q_2}{\left[(\lambda\eta_1+\mu)+\left(2\sqrt{\lambda\eta_1\mu\eta_2}-\lambda\eta_1-\mu\eta_2\right)\theta\right]}a_{n}\\
&+\frac{\lambda\eta_1\mu q_2}{\left[(\lambda\eta_1+\mu)+\left(2\sqrt{\lambda\eta_1\mu\eta_2}-\lambda\eta_1-\mu\eta_2\right)\theta\right]^2}a_{n-1}\\
&+\frac{(\lambda\eta_1)^2\mu q_2}{\left[(\lambda\eta_1+\mu)+\left(2\sqrt{\lambda\eta_1\mu\eta_2}-\lambda\eta_1-\mu\eta_2\right)\theta\right]^3}a_{n-2}\\
&+\cdots\\
&+\frac{(\lambda\eta_1)^{n-2}\mu q_2}{\left[(\lambda\eta_1+\mu)+\left(2\sqrt{\lambda\eta_1\mu\eta_2}-\lambda\eta_1-\mu\eta_2\right)\theta\right]^{n-1}}a_{2}\\
&+\frac{(\lambda\eta_1)^{n-1}\mu q_2}{\left[(\lambda\eta_1+\mu)+\left(2\sqrt{\lambda\eta_1\mu\eta_2}-\lambda\eta_1-\mu\eta_2\right)\theta\right]^{n}}a_{1}\\
&+\frac{\mu\eta_2}{\left[(\lambda\eta_1+\mu)+\left(2\sqrt{\lambda\eta_1\mu\eta_2}-\lambda\eta_1-\mu\eta_2\right)\theta\right]}a_{n+1}\\
&+\frac{\lambda\eta_1\mu\eta_2}{\left[(\lambda\eta_1+\mu)+\left(2\sqrt{\lambda\eta_1\mu\eta_2}-\lambda\eta_1-\mu\eta_2\right)\theta\right]^2}a_{n}\\
&+\frac{(\lambda\eta_1)^2\mu\eta_2}{\left[(\lambda\eta_1+\mu)+\left(2\sqrt{\lambda\eta_1\mu\eta_2}-\lambda\eta_1-\mu\eta_2\right)\theta\right]^3}a_{n-1}\\
&+\cdots\\
&+\frac{(\lambda\eta_1)^{n-1}\mu\eta_2}{\left[(\lambda\eta_1+\mu)+\left(2\sqrt{\lambda\eta_1\mu\eta_2}-\lambda\eta_1-\mu\eta_2\right)\theta\right]^{n}}a_{2}\\
&+\frac{(\lambda\eta_1)^{n}\mu\eta_2}{\left[(\lambda\eta_1+\mu)+\left(2\sqrt{\lambda\eta_1\mu\eta_2}-\lambda\eta_1-\mu\eta_2\right)\theta\right]^{n+1}}a_{1},
\end{aligned}
\tag{4-148}
$$

$$
\begin{aligned}
a_{n+1} &= p_{n+1}(0) \\
&= \mu q_2 \int_0^\infty p_{n+1}(x)dx + \mu\eta_2 \int_0^\infty p_{n+2}(x)dx \\
&= \mu q_2 \int_0^\infty a_{n+1} e^{-\left[(\lambda\eta_1+\mu)+\left(2\sqrt{\lambda\eta_1\mu\eta_2}-\lambda\eta_1-\mu\eta_2\right)\theta\right]x} dx \\
&\quad + \lambda\eta_1\mu q_2 \int_0^\infty e^{-\left[(\lambda\eta_1+\mu)+\left(2\sqrt{\lambda\eta_1\mu\eta_2}-\lambda\eta_1-\mu\eta_2\right)\theta\right]x} \\
&\quad \times \int_0^x e^{\left[(\lambda\eta_1+\mu)+\left(2\sqrt{\lambda\eta_1\mu\eta_2}-\lambda\eta_1-\mu\eta_2\right)\theta\right]\tau} p_n(\tau)d\tau dx \\
&\quad + \mu\eta_2 \int_0^\infty a_{n+2} e^{-\left[(\lambda\eta_1+\mu)+\left(2\sqrt{\lambda\eta_1\mu\eta_2}-\lambda\eta_1-\mu\eta_2\right)\theta\right]x} dx \\
&\quad + \lambda\eta_1\mu\eta_2 \int_0^\infty e^{-\left[(\lambda\eta_1+\mu)+\left(2\sqrt{\lambda\eta_1\mu\eta_2}-\lambda\eta_1-\mu\eta_2\right)\theta\right]x} \\
&\quad \times \int_0^x e^{\left[(\lambda\eta_1+\mu)+\left(2\sqrt{\lambda\eta_1\mu\eta_2}-\lambda\eta_1-\mu\eta_2\right)\theta\right]\tau} p_{n+1}(\tau)d\tau dx \\
&= \frac{\mu q_2}{\left[(\lambda\eta_1+\mu)+\left(2\sqrt{\lambda\eta_1\mu\eta_2}-\lambda\eta_1-\mu\eta_2\right)\theta\right]} a_{n+1} \\
&\quad + \frac{\lambda\eta_1\mu q_2}{\left[(\lambda\eta_1+\mu)+\left(2\sqrt{\lambda\eta_1\mu\eta_2}-\lambda\eta_1-\mu\eta_2\right)\theta\right]} \int_0^\infty p_n(x)dx \\
&\quad + \frac{\mu\eta_2}{\left[(\lambda\eta_1+\mu)+\left(2\sqrt{\lambda\eta_1\mu\eta_2}-\lambda\eta_1-\mu\eta_2\right)\theta\right]} a_{n+2} \\
&\quad + \frac{\lambda\eta_1\mu\eta_2}{\left[(\lambda\eta_1+\mu)+\left(2\sqrt{\lambda\eta_1\mu\eta_2}-\lambda\eta_1-\mu\eta_2\right)\theta\right]} \int_0^\infty p_{n+1}(x)dx \\
&= \cdots \\
&= \frac{\mu q_2}{\left[(\lambda\eta_1+\mu)+\left(2\sqrt{\lambda\eta_1\mu\eta_2}-\lambda\eta_1-\mu\eta_2\right)\theta\right]} a_{n+1} \\
&\quad + \frac{\lambda\eta_1\mu q_2}{\left[(\lambda\eta_1+\mu)+\left(2\sqrt{\lambda\eta_1\mu\eta_2}-\lambda\eta_1-\mu\eta_2\right)\theta\right]^2} a_n \\
&\quad + \frac{(\lambda\eta_1)^2\mu q_2}{\left[(\lambda\eta_1+\mu)+\left(2\sqrt{\lambda\eta_1\mu\eta_2}-\lambda\eta_1-\mu\eta_2\right)\theta\right]^3} a_{n-1} \\
&\quad + \cdots \\
&\quad + \frac{(\lambda\eta_1)^{n-1}\mu q_2}{\left[(\lambda\eta_1+\mu)+\left(2\sqrt{\lambda\eta_1\mu\eta_2}-\lambda\eta_1-\mu\eta_2\right)\theta\right]^n} a_2 \\
&\quad + \frac{(\lambda\eta_1)^n\mu q_2}{\left[(\lambda\eta_1+\mu)+\left(2\sqrt{\lambda\eta_1\mu\eta_2}-\lambda\eta_1-\mu\eta_2\right)\theta\right]^{n+1}} a_1 \\
&\quad + \frac{\mu\eta_2}{\left[(\lambda\eta_1+\mu)+\left(2\sqrt{\lambda\eta_1\mu\eta_2}-\lambda\eta_1-\mu\eta_2\right)\theta\right]} a_{n+2}
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{\lambda\eta_1\mu\eta_2}{\left[(\lambda\eta_1+\mu)+\left(2\sqrt{\lambda\eta_1\mu\eta_2}-\lambda\eta_1-\mu\eta_2\right)\theta\right]^2}a_{n+1}\\
&+\frac{(\lambda\eta_1)^2\mu\eta_2}{\left[(\lambda\eta_1+\mu)+\left(2\sqrt{\lambda\eta_1\mu\eta_2}-\lambda\eta_1-\mu\eta_2\right)\theta\right]^3}a_{n}\\
&+\cdots\\
&+\frac{(\lambda\eta_1)^n\mu\eta_2}{\left[(\lambda\eta_1+\mu)+\left(2\sqrt{\lambda\eta_1\mu\eta_2}-\lambda\eta_1-\mu\eta_2\right)\theta\right]^{n+1}}a_{2}\\
&+\frac{(\lambda\eta_1)^{n+1}\mu\eta_2}{\left[(\lambda\eta_1+\mu)+\left(2\sqrt{\lambda\eta_1\mu\eta_2}-\lambda\eta_1-\mu\eta_2\right)\theta\right]^{n+2}}a_{1}.
\end{aligned}
\tag{4-149}
$$

(4-149)$-\frac{\lambda\eta_1}{[(\lambda\eta_1+\mu)+(2\sqrt{\lambda\eta_1\mu\eta_2}-\lambda\eta_1-\mu\eta_2)\theta]}\times$(4-148) gives

$$
\begin{aligned}
&a_{n+1}-\frac{\lambda\eta_1}{\left[(\lambda\eta_1+\mu)+\left(2\sqrt{\lambda\eta_1\mu\eta_2}-\lambda\eta_1-\mu\eta_2\right)\theta\right]}a_n\\
&=\frac{\mu q_2}{\left[(\lambda\eta_1+\mu)+\left(2\sqrt{\lambda\eta_1\mu\eta_2}-\lambda\eta_1-\mu\eta_2\right)\theta\right]}a_{n+1}\\
&\quad+\frac{\mu\eta_2}{\left[(\lambda\eta_1+\mu)+\left(2\sqrt{\lambda\eta_1\mu\eta_2}-\lambda\eta_1-\mu\eta_2\right)\theta\right]}a_{n+2}\\
&\Longrightarrow\\
a_{n+2}&=\frac{\left[(\lambda\eta_1+\mu)+\left(2\sqrt{\lambda\eta_1\mu\eta_2}-\lambda\eta_1-\mu\eta_2\right)\theta\right]-\mu q_2}{\mu\eta_2}a_{n+1}\\
&\quad-\frac{\lambda\eta_1}{\mu\eta_2}a_n,\quad n\geq 2.
\end{aligned}
\tag{4-150}
$$

Since $\lambda\eta_1<\mu\eta_2$ and $0<\theta<1$, by using $q_2+\eta_2=1$ it follows that

$$
\begin{aligned}
&\lambda\eta_1+\mu+\left(2\sqrt{\lambda\eta_1\mu\eta_2}-\lambda\eta_1-\mu\eta_2\right)\theta-\mu q_2-2\sqrt{\lambda\eta_1\mu\eta_2}\\
&=\lambda\eta_1+\mu\eta_2+\left(2\sqrt{\lambda\eta_1\mu\eta_2}-\lambda\eta_1-\mu\eta_2\right)\theta-2\sqrt{\lambda\eta_1\mu\eta_2}\\
&=\lambda\eta_1+\mu\eta_2-2\sqrt{\lambda\eta_1\mu\eta_2}-\left(\lambda\eta_1+\mu\eta_2-2\sqrt{\lambda\eta_1\mu\eta_2}\right)\theta\\
&=\left(\sqrt{\mu\eta_2}-\sqrt{\lambda\eta_1}\right)^2-\left(\sqrt{\mu\eta_2}-\sqrt{\lambda\eta_1}\right)^2\theta\\
&=\left(\sqrt{\mu\eta_2}-\sqrt{\lambda\eta_1}\right)^2(1-\theta)>0.
\end{aligned}
\tag{4-151}
$$

From this we can rewrite (4-150) as:

$$
a_{n+2}-a_{n+1}\left\{\frac{[(\lambda\eta_1+\mu)+(2\sqrt{\lambda\eta_1\mu\eta_2}-\lambda\eta_1-\mu\eta_2)\theta]-\mu q_2}{2\mu\eta_2}\right.
$$

$$+\frac{\sqrt{([(\lambda\eta_1+\mu)+(2\sqrt{\lambda\eta_1\mu\eta_2}-\lambda\eta_1-\mu\eta_2)\theta]-\mu q_2)^2-4\lambda\eta_1\mu\eta_2}}{2\mu\eta_2}\Bigg\}$$

$$=\Bigg\{\frac{[(\lambda\eta_1+\mu)+(2\sqrt{\lambda\eta_1\mu\eta_2}-\lambda\eta_1-\mu\eta_2)\theta]-\mu q_2}{2\mu\eta_2}$$
$$-\frac{\sqrt{([(\lambda\eta_1+\mu)+(2\sqrt{\lambda\eta_1\mu\eta_2}-\lambda\eta_1-\mu\eta_2)\theta]-\mu q_2)^2-4\lambda\eta_1\mu\eta_2}}{2\mu\eta_2}\Bigg\}$$
$$\times\Bigg\{a_{n+1}-\Bigg[\frac{[(\lambda\eta_1+\mu)+(2\sqrt{\lambda\eta_1\mu\eta_2}-\lambda\eta_1-\mu\eta_2)\theta]-\mu q_2}{2\mu\eta_2}$$
$$+\frac{\sqrt{([(\lambda\eta_1+\mu)+(2\sqrt{\lambda\eta_1\mu\eta_2}-\lambda\eta_1-\mu\eta_2)\theta]-\mu q_2)^2-4\lambda\eta_1\mu\eta_2}}{2\mu\eta_2}\Bigg]a_n\Bigg\},$$
$$n\geq 2. \tag{4-152}$$

If we set

$$\mathbb{y}=\frac{[(\lambda\eta_1+\mu)+(2\sqrt{\lambda\eta_1\mu\eta_2}-\lambda\eta_1-\mu\eta_2)\theta]-\mu q_2}{2\mu\eta_2}$$
$$+\frac{\sqrt{([(\lambda\eta_1+\mu)+(2\sqrt{\lambda\eta_1\mu\eta_2}-\lambda\eta_1-\mu\eta_2)\theta]-\mu q_2)^2-4\lambda\eta_1\mu\eta_2}}{2\mu\eta_2}$$
$$>0.$$

$$\mathbb{z}=\frac{[(\lambda\eta_1+\mu)+(2\sqrt{\lambda\eta_1\mu\eta_2}-\lambda\eta_1-\mu\eta_2)\theta]-\mu q_2}{2\mu\eta_2}$$
$$-\frac{\sqrt{([(\lambda\eta_1+\mu)+(2\sqrt{\lambda\eta_1\mu\eta_2}-\lambda\eta_1-\mu\eta_2)\theta]-\mu q_2)^2-4\lambda\eta_1\mu\eta_2}}{2\mu\eta_2}$$
$$>0,$$

then (4-152) is equivalent to

$$a_{n+2}-\mathbb{y}a_{n+1}=\mathbb{z}(a_{n+1}-\mathbb{y}a_n),\ n\geq 2, \tag{4-153}$$

which gives

$$\begin{aligned}a_{n+2}-\mathbb{y}a_{n+1}&=\mathbb{z}(a_{n+1}-\mathbb{y}a_n)\\&=\mathbb{z}^2(a_n-\mathbb{y}a_{n-1})\\&=\mathbb{z}^3(a_{n-1}-\mathbb{y}a_{n-2})\\&=\cdots\\&=\mathbb{z}^{n-1}(a_3-\mathbb{y}a_2),\quad n\geq 1.\end{aligned} \tag{4-154}$$

If we introduce notations

$$\mathfrak{C} = \begin{pmatrix} -\mathrm{y} & 1 & 0 & 0 & \cdots \\ 0 & -\mathrm{y} & 1 & 0 & \cdots \\ 0 & 0 & -\mathrm{y} & 1 & \cdots \\ 0 & 0 & 0 & -\mathrm{y} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

$$\vec{a} = \begin{pmatrix} a_2 \\ a_3 \\ a_4 \\ a_5 \\ \vdots \end{pmatrix}, \quad \vec{b} = \begin{pmatrix} a_3 - \mathrm{y}a_2 \\ \mathrm{z}(a_3 - \mathrm{y}a_2) \\ \mathrm{z}^2(a_3 - \mathrm{y}a_2) \\ \mathrm{z}^3(a_3 - \mathrm{y}a_2) \\ \vdots \end{pmatrix},$$

$\mathfrak{C}$ is an infinite matrix, $\vec{a}$ and $\vec{b}$ are infinite vectors, then (4-154) is equal to

$$\mathfrak{C}\vec{a} = \vec{b}. \tag{4-155}$$

From the definition of inverse of an infinite matrix it is not difficult to determine

$$\mathfrak{C}^{-1} = \begin{pmatrix} -\frac{1}{\,} & -\frac{1}{{}^{2}} & -\frac{1}{{}^{3}} & -\frac{1}{{}^{4}} & \cdots \\ 0 & -\frac{1}{\,} & -\frac{1}{{}^{2}} & -\frac{1}{{}^{3}} & \cdots \\ 0 & 0 & -\frac{1}{\,} & -\frac{1}{{}^{2}} & \cdots \\ 0 & 0 & 0 & -\frac{1}{\,} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

From this together with (4-155) it is immediate that

$$\begin{aligned} \vec{a} &= \mathfrak{C}^{-1}\vec{b} \\ &\Longrightarrow \\ a_n &= -\frac{1}{\mathrm{y}}\mathrm{z}^{n-2}(a_3 - \mathrm{y}a_2) - \left(\frac{1}{\mathrm{y}}\right)^2 \mathrm{z}^{n-1}(a_3 - \mathrm{y}a_2) \\ &\quad - \left(\frac{1}{\mathrm{y}}\right)^3 \mathrm{z}^{n}(a_3 - \mathrm{y}a_2) - \cdots \\ &= -\sum_{i=1}^{\infty} \left(\frac{1}{\mathrm{y}}\right)^i \mathrm{z}^{n-3+i}(a_3 - \mathrm{y}a_2) \\ &= -\sum_{i=1}^{\infty} \left(\frac{\mathrm{z}}{\mathrm{y}}\right)^i \mathrm{z}^{n-3}(a_3 - \mathrm{y}a_2), \quad n \geq 2 \\ &\Longrightarrow \\ |a_n| &\leq \sum_{i=1}^{\infty} \left(\frac{\mathrm{z}}{\mathrm{y}}\right)^i \mathrm{z}^{n-3}(|a_3| + \mathrm{y}|a_2|), \quad n \geq 2. \end{aligned} \tag{4-156}$$

By combining the definitions of $\mathbb{y}$ and $\mathbb{z}$ with (4-151) it is easy to see

$$0 < \frac{\mathbb{z}}{\mathbb{y}} < 1.$$

This together with (4-156) give

$$\begin{aligned} |a_n| &\leq \sum_{i=1}^{\infty} \left(\frac{\mathbb{z}}{\mathbb{y}}\right)^i \mathbb{z}^{n-3}(|a_3| + \mathbb{y}|a_2|) \\ &= \mathbb{z}^{n-3}\,(|a_3| + \mathbb{y}|a_2|) \sum_{i=1}^{\infty} \left(\frac{\mathbb{z}}{\mathbb{y}}\right)^i \\ &= \frac{1}{\mathbb{y}-\mathbb{z}} \mathbb{z}^{n-2}(|a_3| + \mathbb{y}|a_2|). \end{aligned} \tag{4-157}$$

The condition $\mu\eta_2 > \lambda\eta_1$ and (4-151) imply

$$\begin{aligned} &0 < \mathbb{z} \\ &= \frac{[(\lambda\eta_1 + \mu) + (2\sqrt{\lambda\eta_1\mu\eta_2} - \lambda\eta_1 - \mu\eta_2)\theta] - \mu q_2}{2\mu\eta_2} \\ &- \frac{\sqrt{([(\lambda\eta_1 + \mu) + (2\sqrt{\lambda\eta_1\mu\eta_2} - \lambda\eta_1 - \mu\eta_2)\theta] - \mu q_2)^2 - 4\lambda\eta_1\mu\eta_2}}{2\mu\eta_2} \\ &< \frac{[(\lambda\eta_1 + \mu) + (2\sqrt{\lambda\eta_1\mu\eta_2} - \lambda\eta_1 - \mu\eta_2)\theta] - \mu q_2}{2\mu\eta_2} \\ &= \frac{\lambda\eta_1 + \mu\eta_2 + (2\sqrt{\lambda\eta_1\mu\eta_2} - \lambda\eta_1 - \mu\eta_2)\theta}{2\mu\eta_2} \\ &= \frac{\lambda\eta_1 + \mu\eta_2 - (\sqrt{\lambda\eta_1} - \sqrt{\mu\eta_2})^2\theta}{2\mu\eta_2} \\ &= \frac{1}{2} + \frac{1}{2}\frac{\lambda\eta_1}{\mu\eta_2} - \frac{(\sqrt{\lambda\eta_1} - \sqrt{\mu\eta_2})^2\theta}{2\mu\eta_2} \\ &< \frac{1}{2} + \frac{1}{2}\frac{\lambda\eta_1}{\mu\eta_2} < 1. \end{aligned}$$

From this together with (4-157), (4-144), (4-145) and (4-146) we deduce

$$\begin{aligned} \sum_{n=1}^{\infty} |a_n| &= |a_1| + \sum_{n=2}^{\infty} |a_n| \\ &\leq |a_1| + \sum_{n=2}^{\infty} \frac{1}{\mathbb{y}-\mathbb{z}} \mathbb{z}^{n-2}(|a_3| + \mathbb{y}|a_2|) \\ &= |a_1| + \frac{(|a_3| + \mathbb{y}|a_2|)}{\mathbb{y}-\mathbb{z}} \sum_{n=2}^{\infty} \mathbb{z}^{n-2} \end{aligned}$$

$$= |a_1| + \frac{|a_3| + \mathbb{y}|a_2|}{(1-\mathbb{z})(\mathbb{y}-\mathbb{z})} < \infty. \tag{4-158}$$

By applying (4-143), the Fubini theorem (see Theorem 1.49) and

$$(\lambda\eta_1 + \mu) + \left(2\sqrt{\lambda\eta_1\mu\eta_2} - \lambda\eta_1 - \mu\eta_2\right)\theta > 0$$

we derive

$$\begin{aligned}
\|p_k\|_{L^1[0,\infty)} &= \int_0^\infty |p_k(x)|dx \\
&= \int_0^\infty \Big| a_k e^{-\left[(\lambda\eta_1+\mu)+\left(2\sqrt{\lambda\eta_1\mu\eta_2}-\lambda\eta_1-\mu\eta_2\right)\theta\right]x} \\
&\quad + \lambda\eta_1 e^{-\left[(\lambda\eta_1+\mu)+\left(2\sqrt{\lambda\eta_1\mu\eta_2}-\lambda\eta_1-\mu\eta_2\right)\theta\right]x} \\
&\quad \times \int_0^x e^{\left[(\lambda\eta_1+\mu)+\left(2\sqrt{\lambda\eta_1\mu\eta_2}-\lambda\eta_1-\mu\eta_2\right)\theta\right]\tau} p_{k-1}(\tau)d\tau \Big| dx \\
&\le \int_0^\infty |a_k| e^{-\left[(\lambda\eta_1+\mu)+\left(2\sqrt{\lambda\eta_1\mu\eta_2}-\lambda\eta_1-\mu\eta_2\right)\theta\right]x} dx \\
&\quad + \lambda\eta_1 \int_0^\infty e^{-\left[(\lambda\eta_1+\mu)+\left(2\sqrt{\lambda\eta_1\mu\eta_2}-\lambda\eta_1-\mu\eta_2\right)\theta\right]x} \\
&\quad \times \int_0^x e^{\left[(\lambda\eta_1+\mu)+\left(2\sqrt{\lambda\eta_1\mu\eta_2}-\lambda\eta_1-\mu\eta_2\right)\theta\right]\tau} |p_{k-1}(\tau)| d\tau dx \\
&= \frac{1}{\left[(\lambda\eta_1+\mu)+\left(2\sqrt{\lambda\eta_1\mu\eta_2}-\lambda\eta_1-\mu\eta_2\right)\theta\right]} |a_k| \\
&\quad + \frac{\lambda\eta_1}{\left[(\lambda\eta_1+\mu)+\left(2\sqrt{\lambda\eta_1\mu\eta_2}-\lambda\eta_1-\mu\eta_2\right)\theta\right]} \int_0^\infty |p_{k-1}(\tau)|d\tau \\
&= \frac{1}{\left[(\lambda\eta_1+\mu)+\left(2\sqrt{\lambda\eta_1\mu\eta_2}-\lambda\eta_1-\mu\eta_2\right)\theta\right]} |a_k| \\
&\quad + \frac{\lambda\eta_1}{\left[(\lambda\eta_1+\mu)+\left(2\sqrt{\lambda\eta_1\mu\eta_2}-\lambda\eta_1-\mu\eta_2\right)\theta\right]} \\
&\quad \times \int_0^\infty \Big| a_{k-1} e^{-\left[(\lambda\eta_1+\mu)+\left(2\sqrt{\lambda\eta_1\mu\eta_2}-\lambda\eta_1-\mu\eta_2\right)\theta\right]\tau} \\
&\quad + \lambda\eta_1 e^{-\left[(\lambda\eta_1+\mu)+\left(2\sqrt{\lambda\eta_1\mu\eta_2}-\lambda\eta_1-\mu\eta_2\right)\theta\right]\tau} \\
&\quad \times \int_0^\tau e^{\left[(\lambda\eta_1+\mu)+\left(2\sqrt{\lambda\eta_1\mu\eta_2}-\lambda\eta_1-\mu\eta_2\right)\theta\right]\xi} p_{k-2}(\xi)d\xi \Big| d\tau \\
&\le \frac{1}{\left[(\lambda\eta_1+\mu)+\left(2\sqrt{\lambda\eta_1\mu\eta_2}-\lambda\eta_1-\mu\eta_2\right)\theta\right]} |a_k| \\
&\quad + \frac{\lambda\eta_1}{\left[(\lambda\eta_1+\mu)+\left(2\sqrt{\lambda\eta_1\mu\eta_2}-\lambda\eta_1-\mu\eta_2\right)\theta\right]}
\end{aligned}$$

$$\times \int_0^\infty |a_{k-1}| e^{-\left[(\lambda\eta_1+\mu)+\left(2\sqrt{\lambda\eta_1\mu\eta_2}-\lambda\eta_1-\mu\eta_2\right)\theta\right]\tau} d\tau$$
$$+ \frac{(\lambda\eta_1)^2}{\left[(\lambda\eta_1+\mu)+\left(2\sqrt{\lambda\eta_1\mu\eta_2}-\lambda\eta_1-\mu\eta_2\right)\theta\right]}$$
$$\times \int_0^\infty e^{-\left[(\lambda\eta_1+\mu)+\left(2\sqrt{\lambda\eta_1\mu\eta_2}-\lambda\eta_1-\mu\eta_2\right)\theta\right]\tau}$$
$$\times \int_0^\tau |p_{k-2}(\xi)| e^{\left[(\lambda\eta_1+\mu)+\left(2\sqrt{\lambda\eta_1\mu\eta_2}-\lambda\eta_1-\mu\eta_2\right)\theta\right]\xi} d\xi d\tau$$

$$= \frac{1}{\left[(\lambda\eta_1+\mu)+\left(2\sqrt{\lambda\eta_1\mu\eta_2}-\lambda\eta_1-\mu\eta_2\right)\theta\right]} |a_k|$$
$$+ \frac{\lambda\eta_1}{\left[(\lambda\eta_1+\mu)+\left(2\sqrt{\lambda\eta_1\mu\eta_2}-\lambda\eta_1-\mu\eta_2\right)\theta\right]^2} |a_{k-1}|$$
$$+ \frac{(\lambda\eta_1)^2}{\left[(\lambda\eta_1+\mu)+\left(2\sqrt{\lambda\eta_1\mu\eta_2}-\lambda\eta_1-\mu\eta_2\right)\theta\right]}$$
$$\times \int_0^\infty |p_{k-2}(\xi)| e^{\left[(\lambda\eta_1+\mu)+\left(2\sqrt{\lambda\eta_1\mu\eta_2}-\lambda\eta_1-\mu\eta_2\right)\theta\right]\xi}$$
$$\times \int_\xi^\infty e^{-\left[(\lambda\eta_1+\mu)+\left(2\sqrt{\lambda\eta_1\mu\eta_2}-\lambda\eta_1-\mu\eta_2\right)\theta\right]\tau} d\tau d\xi$$

$$= \frac{1}{\left[(\lambda\eta_1+\mu)+\left(2\sqrt{\lambda\eta_1\mu\eta_2}-\lambda\eta_1-\mu\eta_2\right)\theta\right]} |a_k|$$
$$+ \frac{\lambda\eta_1}{\left[(\lambda\eta_1+\mu)+\left(2\sqrt{\lambda\eta_1\mu\eta_2}-\lambda\eta_1-\mu\eta_2\right)\theta\right]^2} |a_{k-1}|$$
$$+ \frac{(\lambda\eta_1)^2}{\left[(\lambda\eta_1+\mu)+\left(2\sqrt{\lambda\eta_1\mu\eta_2}-\lambda\eta_1-\mu\eta_2\right)\theta\right]^2} \int_0^\infty |p_{k-2}(\xi)| d\xi$$

$$= \frac{1}{\left[(\lambda\eta_1+\mu)+\left(2\sqrt{\lambda\eta_1\mu\eta_2}-\lambda\eta_1-\mu\eta_2\right)\theta\right]} |a_k|$$
$$+ \frac{\lambda\eta_1}{\left[(\lambda\eta_1+\mu)+\left(2\sqrt{\lambda\eta_1\mu\eta_2}-\lambda\eta_1-\mu\eta_2\right)\theta\right]^2} |a_{k-1}|$$
$$+ \frac{(\lambda\eta_1)^2}{\left[(\lambda\eta_1+\mu)+\left(2\sqrt{\lambda\eta_1\mu\eta_2}-\lambda\eta_1-\mu\eta_2\right)\theta\right]^2}$$
$$\times \int_0^\infty \Big| a_{k-2} e^{-\left[(\lambda\eta_1+\mu)+\left(2\sqrt{\lambda\eta_1\mu\eta_2}-\lambda\eta_1-\mu\eta_2\right)\theta\right]\xi}$$
$$+ \lambda\eta_1 e^{-\left[(\lambda\eta_1+\mu)+\left(2\sqrt{\lambda\eta_1\mu\eta_2}-\lambda\eta_1-\mu\eta_2\right)\theta\right]\xi}$$
$$\times \int_0^\xi e^{\left[(\lambda\eta_1+\mu)+\left(2\sqrt{\lambda\eta_1\mu\eta_2}-\lambda\eta_1-\mu\eta_2\right)\theta\right]\tau} p_{k-3}(\tau) d\tau \Big| d\xi$$

$$
\begin{aligned}
&\le \frac{1}{\left[(\lambda\eta_1+\mu)+\left(2\sqrt{\lambda\eta_1\mu\eta_2}-\lambda\eta_1-\mu\eta_2\right)\theta\right]}|a_k| \\
&\quad + \frac{\lambda\eta_1}{\left[(\lambda\eta_1+\mu)+\left(2\sqrt{\lambda\eta_1\mu\eta_2}-\lambda\eta_1-\mu\eta_2\right)\theta\right]^2}|a_{k-1}| \\
&\quad + \frac{(\lambda\eta_1)^2}{\left[(\lambda\eta_1+\mu)+\left(2\sqrt{\lambda\eta_1\mu\eta_2}-\lambda\eta_1-\mu\eta_2\right)\theta\right]^3}|a_{k-2}| \\
&\quad + \frac{(\lambda\eta_1)^3}{\left[(\lambda\eta_1+\mu)+\left(2\sqrt{\lambda\eta_1\mu\eta_2}-\lambda\eta_1-\mu\eta_2\right)\theta\right]^2} \\
&\quad \times \int_0^\infty e^{-\left[(\lambda\eta_1+\mu)+\left(2\sqrt{\lambda\eta_1\mu\eta_2}-\lambda\eta_1-\mu\eta_2\right)\theta\right]\xi} \\
&\quad \times \int_0^\xi |p_{k-3}(\tau)|e^{\left[(\lambda\eta_1+\mu)+\left(2\sqrt{\lambda\eta_1\mu\eta_2}-\lambda\eta_1-\mu\eta_2\right)\theta\right]^\tau} d\tau d\xi \\
&= \frac{1}{\left[(\lambda\eta_1+\mu)+\left(2\sqrt{\lambda\eta_1\mu\eta_2}-\lambda\eta_1-\mu\eta_2\right)\theta\right]}|a_k| \\
&\quad + \frac{\lambda\eta_1}{\left[(\lambda\eta_1+\mu)+\left(2\sqrt{\lambda\eta_1\mu\eta_2}-\lambda\eta_1-\mu\eta_2\right)\theta\right]^2}|a_{k-1}| \\
&\quad + \frac{(\lambda\eta_1)^2}{\left[(\lambda\eta_1+\mu)+\left(2\sqrt{\lambda\eta_1\mu\eta_2}-\lambda\eta_1-\mu\eta_2\right)\theta\right]^3}|a_{k-2}| \\
&\quad + \frac{(\lambda\eta_1)^3}{\left[(\lambda\eta_1+\mu)+\left(2\sqrt{\lambda\eta_1\mu\eta_2}-\lambda\eta_1-\mu\eta_2\right)\theta\right]^2} \\
&\quad \times \int_0^\infty |p_{k-3}(\tau)|e^{\left[(\lambda\eta_1+\mu)+\left(2\sqrt{\lambda\eta_1\mu\eta_2}-\lambda\eta_1-\mu\eta_2\right)\theta\right]\tau} \\
&\quad \times \int_\tau^\infty e^{-\left[(\lambda\eta_1+\mu)+\left(2\sqrt{\lambda\eta_1\mu\eta_2}-\lambda\eta_1-\mu\eta_2\right)\theta\right]\xi} d\xi d\tau \\
&= \frac{1}{\left[(\lambda\eta_1+\mu)+\left(2\sqrt{\lambda\eta_1\mu\eta_2}-\lambda\eta_1-\mu\eta_2\right)\theta\right]}|a_k| \\
&\quad + \frac{\lambda\eta_1}{\left[(\lambda\eta_1+\mu)+\left(2\sqrt{\lambda\eta_1\mu\eta_2}-\lambda\eta_1-\mu\eta_2\right)\theta\right]^2}|a_{k-1}| \\
&\quad + \frac{(\lambda\eta_1)^2}{\left[(\lambda\eta_1+\mu)+\left(2\sqrt{\lambda\eta_1\mu\eta_2}-\lambda\eta_1-\mu\eta_2\right)\theta\right]^3}|a_{k-2}| \\
&\quad + \frac{(\lambda\eta_1)^3}{\left[(\lambda\eta_1+\mu)+\left(2\sqrt{\lambda\eta_1\mu\eta_2}-\lambda\eta_1-\mu\eta_2\right)\theta\right]^3} \int_0^\infty |p_{k-3}(\tau)| d\tau \\
&\le \cdots \\
&\le \frac{1}{\left[(\lambda\eta_1+\mu)+\left(2\sqrt{\lambda\eta_1\mu\eta_2}-\lambda\eta_1-\mu\eta_2\right)\theta\right]}|a_k|
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{\lambda\eta_1}{\left[(\lambda\eta_1+\mu)+\left(2\sqrt{\lambda\eta_1\mu\eta_2}-\lambda\eta_1-\mu\eta_2\right)\theta\right]^2}|a_{k-1}| \\
&+\frac{(\lambda\eta_1)^2}{\left[(\lambda\eta_1+\mu)+\left(2\sqrt{\lambda\eta_1\mu\eta_2}-\lambda\eta_1-\mu\eta_2\right)\theta\right]^3}|a_{k-2}| \\
&+\cdots \\
&+\frac{(\lambda\eta_1)^{k-3}}{\left[(\lambda\eta_1+\mu)+\left(2\sqrt{\lambda\eta_1\mu\eta_2}-\lambda\eta_1-\mu\eta_2\right)\theta\right]^{k-2}}|a_3| \\
&+\frac{(\lambda\eta_1)^{k-2}}{\left[(\lambda\eta_1+\mu)+\left(2\sqrt{\lambda\eta_1\mu\eta_2}-\lambda\eta_1-\mu\eta_2\right)\theta\right]^{k-1}}|a_2| \\
&+\frac{(\lambda\eta_1)^{k-1}}{\left[(\lambda\eta_1+\mu)+\left(2\sqrt{\lambda\eta_1\mu\eta_2}-\lambda\eta_1-\mu\eta_2\right)\theta\right]^{k}}|a_1| \\
=&\frac{1}{\left[(\lambda\eta_1+\mu)+\left(2\sqrt{\lambda\eta_1\mu\eta_2}-\lambda\eta_1-\mu\eta_2\right)\theta\right]} \\
&\times\sum_{j=0}^{k-1}\left(\frac{\lambda\eta_1}{\left[(\lambda\eta_1+\mu)+\left(2\sqrt{\lambda\eta_1\mu\eta_2}-\lambda\eta_1-\mu\eta_2\right)\theta\right]}\right)^j|a_{k-j}|, \\
&k\geq 2, \qquad\qquad (4\text{-}159)
\end{aligned}
$$

and (4-142), (4-144) and (4-159) give

$$
\begin{aligned}
&\sum_{k=1}^{n}\|p_k\|_{L^1[0,\infty)}=\|p_1\|_{L^1[0,\infty)}+\sum_{k=2}^{n}\|p_k\|_{L^1[0,\infty)} \\
&\leq|a_1|\int_0^\infty e^{-\left[(\lambda\eta_1+\mu)+\left(2\sqrt{\lambda\eta_1\mu\eta_2}-\lambda\eta_1-\mu\eta_2\right)\theta\right]x}dx \\
&\quad+\frac{1}{\left[(\lambda\eta_1+\mu)+\left(2\sqrt{\lambda\eta_1\mu\eta_2}-\lambda\eta_1-\mu\eta_2\right)\theta\right]} \\
&\quad\times\sum_{k=2}^{n}\sum_{j=0}^{k-1}\left(\frac{\lambda\eta_1}{\left[(\lambda\eta_1+\mu)+\left(2\sqrt{\lambda\eta_1\mu\eta_2}-\lambda\eta_1-\mu\eta_2\right)\theta\right]}\right)^j|a_{k-j}| \\
&\leq\frac{1}{\left[(\lambda\eta_1+\mu)+\left(2\sqrt{\lambda\eta_1\mu\eta_2}-\lambda\eta_1-\mu\eta_2\right)\theta\right]}|a_1| \\
&\quad+\frac{1}{\left[(\lambda\eta_1+\mu)+\left(2\sqrt{\lambda\eta_1\mu\eta_2}-\lambda\eta_1-\mu\eta_2\right)\theta\right]} \\
&\quad\times\sum_{k=2}^{n}\sum_{j=0}^{k-1}\left(\frac{\lambda\eta_1}{\left[(\lambda\eta_1+\mu)+\left(2\sqrt{\lambda\eta_1\mu\eta_2}-\lambda\eta_1-\mu\eta_2\right)\theta\right]}\right)^j|a_{k-j}| \\
&=\frac{\lambda\eta_1+\left(2\sqrt{\lambda\eta_1\mu\eta_2}-\lambda\eta_1-\mu\eta_2\right)\theta}{\mu\eta_2}|p_0|
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{1}{\left[(\lambda\eta_1+\mu)+\left(2\sqrt{\lambda\eta_1\mu\eta_2}-\lambda\eta_1-\mu\eta_2\right)\theta\right]}\\
&\times\sum_{j=0}^{n-1}\sum_{k=j+1}^{n}\left(\frac{\lambda\eta_1}{\left[(\lambda\eta_1+\mu)+\left(2\sqrt{\lambda\eta_1\mu\eta_2}-\lambda\eta_1-\mu\eta_2\right)\theta\right]}\right)^j|a_{k-j}|\\
&=\frac{\lambda\eta_1+\left(2\sqrt{\lambda\eta_1\mu\eta_2}-\lambda\eta_1-\mu\eta_2\right)\theta}{\mu\eta_2}|p_0|\\
&\quad+\frac{1}{\left[(\lambda\eta_1+\mu)+\left(2\sqrt{\lambda\eta_1\mu\eta_2}-\lambda\eta_1-\mu\eta_2\right)\theta\right]}\\
&\quad\times\sum_{j=0}^{n-1}\sum_{l=1}^{n-j}\left(\frac{\lambda\eta_1}{\left[(\lambda\eta_1+\mu)+\left(2\sqrt{\lambda\eta_1\mu\eta_2}-\lambda\eta_1-\mu\eta_2\right)\theta\right]}\right)^j|a_l|\\
&\le\frac{\lambda\eta_1+\left(2\sqrt{\lambda\eta_1\mu\eta_2}-\lambda\eta_1-\mu\eta_2\right)\theta}{\mu\eta_2}|p_0|\\
&\quad+\frac{1}{\left[(\lambda\eta_1+\mu)+\left(2\sqrt{\lambda\eta_1\mu\eta_2}-\lambda\eta_1-\mu\eta_2\right)\theta\right]}\\
&\quad\times\sum_{j=0}^{n-1}\left(\frac{\lambda\eta_1}{[(\lambda\eta_1+\mu)+(2\sqrt{\lambda\eta_1\mu\eta_2}-\lambda\eta_1-\mu\eta_2)\theta]}\right)^j\sum_{l=1}^{n}|a_l|. \qquad (4\text{-}160)
\end{aligned}
$$

From the conditions $\mu\eta_2>\lambda\eta_1$ and $\lambda>0,\ \eta_1\in(0,1),\ \eta_2\in(0,1),\ \theta\in(0,1)$ it follows that

$$
\begin{aligned}
&(1-\eta_2\theta)\mu+\lambda\eta_1\theta>0\\
&\Longrightarrow\\
&(1-\eta_2\theta)\mu+(2\lambda\eta_1-\lambda\eta_1)\theta>0\\
&\Longrightarrow\\
&0<(1-\eta_2\theta)\mu+\left(2\sqrt{\lambda\eta_1\lambda\eta_1}-\lambda\eta_1\right)\theta\\
&\ <(1-\eta_2\theta)\mu+\left(2\sqrt{\lambda\eta_1\mu\eta_2}-\lambda\eta_1\right)\theta\\
&\Longrightarrow\\
&(1-\eta_2\theta)\mu+\left(2\sqrt{\lambda\eta_1\mu\eta_2}-\lambda\eta_1\right)\theta>0\\
&\Longrightarrow\\
&\mu-\mu\eta_2\theta+\left(2\sqrt{\lambda\eta_1\mu\eta_2}-\lambda\eta_1\right)\theta>0\\
&\Longrightarrow\\
&\mu+\left(2\sqrt{\lambda\eta_1\mu\eta_2}-\lambda\eta_1-\mu\eta_2\right)\theta>0\\
&\Longrightarrow
\end{aligned}
$$

$$\begin{aligned}
&\lambda\eta_1 < \lambda\eta_1 + \mu + \left(2\sqrt{\lambda\eta_1\mu\eta_2} - \lambda\eta_1 - \mu\eta_2\right)\theta \\
&\Longrightarrow \\
&\frac{\lambda\eta_1}{\lambda\eta_1 + \mu + \left(2\sqrt{\lambda\eta_1\mu\eta_2} - \lambda\eta_1 - \mu\eta_2\right)\theta} < 1.
\end{aligned}$$

This together with (4-158), (4-159) and (4-160) give

$$\begin{aligned}
\|p\| &= |p_0| + \lim_{n\to\infty}\sum_{k=1}^{n}\|p_k\|_{L^1[0,\infty)} \\
&\le |p_0| + \frac{\lambda\eta_1 + \left(2\sqrt{\lambda\eta_1\mu\eta_2} - \lambda\eta_1 - \mu\eta_2\right)\theta}{\mu\eta_2}|p_0| \\
&\quad + \frac{1}{[(\lambda\eta_1+\mu) + (2\sqrt{\lambda\eta_1\mu\eta_2} - \lambda\eta_1 - \mu\eta_2)\theta]} \\
&\quad \times \sum_{j=0}^{\infty}\left(\frac{\lambda\eta_1}{\left[(\lambda\eta_1+\mu) + \left(2\sqrt{\lambda\eta_1\mu\eta_2} - \lambda\eta_1 - \mu\eta_2\right)\theta\right]}\right)^j \sum_{l=1}^{\infty}|a_l| \\
&= |p_0| + \frac{\lambda\eta_1 + \left(2\sqrt{\lambda\eta_1\mu\eta_2} - \lambda\eta_1 - \mu\eta_2\right)\theta}{\mu\eta_2}|p_0| \\
&\quad + \frac{1}{\left[(\lambda\eta_1+\mu) + \left(2\sqrt{\lambda\eta_1\mu\eta_2} - \lambda\eta_1 - \mu\eta_2\right)\theta\right] - \lambda\eta_1}\sum_{l=1}^{\infty}|a_l| \\
&= |p_0| + \frac{\lambda\eta_1 + \left(2\sqrt{\lambda\eta_1\mu\eta_2} - \lambda\eta_1 - \mu\eta_2\right)\theta}{\mu\eta_2}|p_0| \\
&\quad + \frac{1}{\mu + \left(2\sqrt{\lambda\eta_1\mu\eta_2} - \lambda\eta_1 - \mu\eta_2\right)\theta}\sum_{l=1}^{\infty}|a_l| \\
&< \infty.
\end{aligned}$$

This shows that $\left(2\sqrt{\lambda\eta_1\mu\eta_2} - \lambda\eta_1 - \mu\eta_2\right)\theta$ are eigenvalues of $A+U+E$ for all $0<\theta<1$ (see Definition 1.22). Moreover, by (4-142), (4-143), (4-144), (4-145), (4-146) and (4-150) it is easy to see that the eigenvectors corresponding to each $(2\sqrt{\lambda\eta_1\mu\eta_2} - \lambda\eta_1 - \mu\eta_2)\theta$ form a one-dimensional linear space, that is, the geometric multiplicity of each $\left(2\sqrt{\lambda\eta_1\mu\eta_2} - \lambda\eta_1 - \mu\eta_2\right)\theta$ is 1 (see Definition 1.23). $\square$

Theorem 4.11 shows that $A+U+E$ has uncountable many eigenvalues in the left half complex plane and therefore $T(t)$ is not compact, even not eventually compact. Moreover, this theorem shows that 0 is not an isolated singular point of $(\gamma I - A - U - E)^{-1}$. Hence, the result obtained in Theorem 4.10 is the best result about asymptotic behavior of the time-dependent solution of the system (4-23), that is to say, it is impossible that the time-dependent solution of the system (4-23) exponentially converges to its steady-state solution.

4.4 Asymptotic Behavior of the Time-dependent Solution of the System (4-23)

According to Theorem 1.96, if we know the spectrum of $A+U+E$ on the imaginary axis, by using Theorem 4.2 we immediately get the asymptotic behavior of the time-dependent solution of the system (4-23). In fact, through investigating the system (4-23) it is easy to see that the difficult point is the boundary conditions. And G. Greiner [45] promoted an idea through which the original system can be simplified by perturbing the boundary conditions and the desired result can be deduced by discussing the boundary operator. In 2006, Agnes Radl [97] successfully applied Greiner's idea to study the M/$M^{k,B}$/1 queueing model and a reliability model which was described by a system of finite partial differential equations with integral boundary conditions. By applying a result in Radl [97] to the system (4-23) we obtain that all points on the imaginary axis except for zero belong to the resolvent set of $A+U+E$, and through introducing the probability generating function and discussing it we will show that 0 is an eigenvalue of $A+U+E$ with geometric multiplicity 1. Thus, by using Theorem 1.96 we derive the main result in this section. In order to do this, we first introduce the maximal operator A_m, the boundary operators L, Φ and their domains. Next we rewrite the equations (4-16)–(4-21) as the system (4-23) by A_m, L and Φ. Thirdly, by using the probability generating function we will verify that 0 is an eigenvalue of $A+U+E$ with geometric multiplicity 1. Fourthly, we define the operator A_0 and its domain, then we will discuss the resolvent set of A_0. Finally, we study the kernel of $\gamma I - A_m$, through which we introduce the Dirichlet operator D_γ and determine the expression of ΦD_γ, by discussing the spectral radius of which and using the result in Radl [97] we obtain the resolvent set of $A+U+E$.

The maximal operator $(A_m, D(A_m))$ (see Greiner [45]) isdefined as

$$A_m \begin{pmatrix} p_0 \\ p_1(x) \\ p_2(x) \\ p_3(x) \\ \vdots \end{pmatrix} = \begin{pmatrix} -\lambda\eta_1 & \eta_2\varphi & 0 & 0 & \cdots \\ 0 & \psi & 0 & 0 & \cdots \\ 0 & \lambda\eta_1 & \psi & 0 & \cdots \\ 0 & 0 & \lambda\eta_1 & \psi & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} p_0 \\ p_1(x) \\ p_2(x) \\ p_3(x) \\ \vdots \end{pmatrix},$$

$$D(A_m) = \left\{ p \in X \;\middle|\; \begin{array}{l} \frac{dp_n}{dx} \in L^1[0,\infty), p_n(x)\ (n \geq 1) \text{ are} \\ \text{absolutely continuous functions} \\ \text{and } \sum\limits_{n=1}^{\infty} \left\| \frac{dp_n}{dx} \right\|_{L^1[0,\infty)} < \infty \end{array} \right\},$$

where

$$\varphi : L^1[0,\infty) \to \mathbb{C}, \quad \varphi(f) := \int_0^\infty \mu(x) f(x) dx, \quad \forall f \in L^1[0,\infty).$$

$$\psi f := -\frac{d}{dx} f - (\lambda\eta_1 + \mu(x)) f, \quad \forall f \in D(A_m).$$

It is easy to see that the operator $(A_m, D(A_m))$ is closed on X. We take the boundary space of X as

$$\partial X := l^1$$

and define boundary operators as

$$L : D(A_m) \to \partial X, \quad L \begin{pmatrix} p_0 \\ p_1(x) \\ p_2(x) \\ p_3(x) \\ \vdots \end{pmatrix} = \begin{pmatrix} p_1(0) \\ p_2(0) \\ p_3(0) \\ \vdots \end{pmatrix}, \quad \forall p \in D(A_m),$$

and

$$\Phi : D(A_m) \to \partial X,$$

$$\Phi \begin{pmatrix} p_0 \\ p_1(x) \\ p_2(x) \\ p_3(x) \\ \vdots \end{pmatrix} := \begin{pmatrix} \lambda\eta_1 & q_2\varphi & \eta_2\varphi & 0 & 0 & 0 & \cdots \\ 0 & 0 & q_2\varphi & \eta_2\varphi & 0 & 0 & \cdots \\ 0 & 0 & 0 & q_2\varphi & \eta_2\varphi & 0 & \cdots \\ 0 & 0 & 0 & 0 & q_2\varphi & \eta_2\varphi & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} p_0 \\ p_1(x) \\ p_2(x) \\ p_3(x) \\ \vdots \end{pmatrix}.$$

If we define the operator $(A + U + E, D(A + U + E))$ by

$$\begin{aligned} (A + U + E)p &:= A_m p, \\ D(A + U + E) &:= \{p \in D(A_m) \mid Lp = \Phi p\}, \end{aligned}$$

then the above system of equations (4-16)–(4-21) can be expressed as the abstract Cauchy problem (4-23).

Lemma 4.12. *If $\eta_2 > \lambda\eta_1 \int_0^\infty e^{-\int_0^x \mu(\tau)d\tau} dx$, then 0 is an eigenvalue of $A+U+E$ with geometric multiplicity one.*

Proof. We consider the equation $(A + U + E)p = 0$, i.e.,

$$\lambda\eta_1 p_0 = \eta_2 \int_0^\infty p_1(x)\mu(x)dx, \tag{4-161}$$

$$\frac{dp_1(x)}{dx} = -(\lambda\eta_1 + \mu(x))p_1(x), \tag{4-162}$$

$$\frac{dp_n(x)}{dx} = -(\lambda\eta_1 + \mu(x))p_n(x) + \lambda\eta_1 p_{n-1}(x), \ n \geq 2, \tag{4-163}$$

$$p_1(0) = \lambda\eta_1 p_0 + q_2 \int_0^\infty p_1(x)\mu(x)dx + \eta_2 \int_0^\infty p_2(x)\mu(x)dx, \tag{4-164}$$

$$p_n(0) = q_2 \int_0^\infty p_n(x)\mu(x)dx + \eta_2 \int_0^\infty p_{n+1}(x)\mu(x)dx, \quad n \geq 2. \tag{4-165}$$

By solving (4-162) and (4-163) we have

$$p_1(x) = a_1 e^{-\lambda\eta_1 x - \int_0^x \mu(\tau)d\tau}, \tag{4-166}$$

$$\begin{aligned} p_n(x) &= a_n e^{-\lambda\eta_1 x - \int_0^x \mu(\tau)d\tau} + \lambda\eta_1 e^{-\lambda\eta_1 x - \int_0^x \mu(\xi)d\xi} \\ &\quad \times \int_0^x p_{n-1}(\tau) e^{\lambda\eta_1\tau + \int_0^\tau \mu(\xi)d\xi} d\tau, \quad n \geq 2. \end{aligned} \tag{4-167}$$

Through using (4-166) and (4-167) repeatedly we deduce

$$p_n(x) = e^{-\lambda\eta_1 x - \int_0^x \mu(\tau)d\tau} \sum_{k=0}^{n-1} \frac{(\lambda\eta_1 x)^k}{k!} a_{n-k}, \quad n \geq 2. \tag{4-168}$$

By combining (4-166) with (4-161) and (4-168) with (4-164) and (4-165) we obtain

$$a_1 = \frac{\lambda\eta_1 p_0}{\eta_2 \int_0^\infty \mu(x) e^{-\lambda\eta_1 x - \int_0^x \mu(\tau)d\tau} dx}. \tag{4-169}$$

$$\begin{aligned} a_n &= q_2 \int_0^\infty \mu(x) e^{-\lambda\eta_1 x - \int_0^x \mu(\tau)d\tau} \sum_{k=0}^{n-1} \frac{(\lambda\eta_1 x)^k}{k!} a_{n-k} dx \\ &\quad + \eta_2 \int_0^\infty \mu(x) e^{-\lambda\eta_1 x - \int_0^x \mu(\tau)d\tau} \\ &\quad \times \sum_{k=0}^{n} \frac{(\lambda\eta_1 x)^k}{k!} a_{n+1-k} dx, \quad n \geq 2. \end{aligned} \tag{4-170}$$

In order to estimate the norm of $p(x)$ we introduce the probability generating function $p(x, z) = \sum_{n=1}^{\infty} p_n(x) z^n$ for all complex variables $|z| < 1$. And Theorem 4.3 implies that $p(x, z)$ is well defined. (4-162) and (4-163) give

$$\frac{\partial \sum_{n=1}^{\infty} p_n(x) z^n}{\partial x} = -\sum_{n=1}^{\infty} (\lambda\eta_1 + \mu(x)) p_n(x) z^n + \sum_{n=2}^{\infty} \lambda\eta_1 p_{n-1}(x) z^n$$

$$\Longrightarrow$$

$$\frac{\partial p(x, z)}{\partial x} = -(\lambda\eta_1 + \mu(x)) p(x, z) + \lambda\eta_1 z \sum_{n=2}^{\infty} p_{n-1}(x) z^{n-1}$$

$$\Longrightarrow$$

$$\begin{aligned} \frac{\partial p(x, z)}{\partial x} &= -(\lambda\eta_1 + \mu(x)) p(x, z) + \lambda\eta_1 z p(x, z) \\ &= (\lambda\eta_1 z - \lambda\eta_1 - \mu(x)) p(x, z) \end{aligned}$$

$$\Longrightarrow$$

$$p(x, z) = p(0, z) e^{\lambda\eta_1(z-1)x - \int_0^x \mu(\tau)d\tau}. \tag{4-171}$$

From this together with (4-161), (4-164), (4-165) and (4-171) it follows that

$$
\begin{aligned}
p(0,z) &= \sum_{n=1}^{\infty} p_n(0)z^n = p_1(0)z + \sum_{n=2}^{\infty} p_n(0)z^n \\
&= \lambda\eta_1 p_0 z + q_2 \int_0^{\infty} \sum_{n=1}^{\infty} p_n(x)\mu(x)z^n dx \\
&\quad + \eta_2 \int_0^{\infty} \sum_{n=1}^{\infty} p_{n+1}(x)\mu(x)z^n dx \\
&= \lambda\eta_1 p_0 z + q_2 \int_0^{\infty} \mu(x)p(x,z)dx \\
&\quad + \frac{\eta_2}{z} \int_0^{\infty} \mu(x) \sum_{n=1}^{\infty} p_{n+1}(x)z^{n+1} dx \\
&= \lambda\eta_1 p_0 z + q_2 \int_0^{\infty} \mu(x)p(x,z)dx \\
&\quad + \frac{\eta_2}{z} \int_0^{\infty} \mu(x) \left(\sum_{n=0}^{\infty} p_{n+1}(x)z^{n+1} - p_1(x)z \right) dx \\
&= \lambda\eta_1 p_0 z + q_2 \int_0^{\infty} \mu(x)p(x,z)dx \\
&\quad + \frac{\eta_2}{z} \int_0^{\infty} \mu(x)p(x,z)dx - \eta_2 \int_0^{\infty} \mu(x)p_1(x)dx \\
&= \left(q_2 + \frac{\eta_2}{z}\right) \int_0^{\infty} \mu(x)p(x,z)dx + \lambda\eta_1 p_0(z-1) \\
&= \left(q_2 + \frac{\eta_2}{z}\right) \int_0^{\infty} \mu(x)p(0,z)e^{\lambda\eta_1(z-1)x - \int_0^x \mu(\tau)d\tau} dx \\
&\quad + \lambda\eta_1 p_0(z-1) \\
&= \left(q_2 + \frac{\eta_2}{z}\right) p(0,z) \int_0^{\infty} \mu(x)e^{\lambda\eta_1(z-1)x - \int_0^x \mu(\tau)d\tau} dx \\
&\quad + \lambda\eta_1 p_0(z-1) \\
&\Longrightarrow
\end{aligned}
$$

$$
p(0,z) = \frac{\lambda\eta_1 p_0(z-1)}{1 - \left(q_2 + \frac{\eta_2}{z}\right) \int_0^{\infty} \mu(x)e^{\lambda\eta_1(z-1)x - \int_0^x \mu(\tau)d\tau} dx}. \tag{4-172}
$$

By inserting (4-172) into (4-171) and using the de l'Hôpital rule and $\eta_2 + q_2 = 1$ and integration by parts we derive (take $p_0 > 0$)

$$
\begin{aligned}
p(x,z) = &\frac{\lambda\eta_1 p_0(z-1)}{1 - \left(q_2 + \frac{\eta_2}{z}\right) \int_0^{\infty} \mu(x)e^{\lambda\eta_1(z-1)x - \int_0^x \mu(\tau)d\tau} dx} \\
&\times e^{\lambda\eta_1(z-1)x - \int_0^x \mu(\tau)d\tau}
\end{aligned}
$$

$$\begin{aligned}
&\Longrightarrow \\
&\sum_{n=1}^{\infty} p_n(x) = \lim_{z\to 1} p(x,z) \\
&= \lim_{z\to 1}\{\lambda\eta_1 p_0\} \Big/ \Bigg\{ \frac{\eta_2}{z^2}\int_0^\infty \mu(x) e^{\lambda\eta_1(z-1)x - \int_0^x \mu(\tau)d\tau} dx \\
&\quad - \left(q_2 + \frac{\eta_2}{z}\right)\int_0^\infty \lambda\eta_1 x\mu(x) e^{\lambda\eta_1(z-1)x - \int_0^x \mu(\tau)d\tau} dx \Bigg\} \\
&\quad \times \lim_{z\to 1} e^{\lambda\eta_1(z-1)x - \int_0^x \mu(\tau)d\tau} \\
&= \frac{\lambda\eta_1 p_0 e^{-\int_0^x \mu(\tau)d\tau}}{\eta_2 \int_0^\infty \mu(x) e^{-\int_0^x \mu(\tau)d\tau} dx + (q_2+\eta_2)\lambda\eta_1 \int_0^\infty x de^{-\int_0^x \mu(\tau)d\tau}} \\
&= \frac{\lambda\eta_1 p_0 e^{-\int_0^x \mu(\tau)d\tau}}{\eta_2 - \lambda\eta_1 \int_0^\infty e^{-\int_0^x \mu(\tau)d\tau} dx} \\
&\Longrightarrow
\end{aligned}$$

$$\begin{aligned}
\|p\| &= |p_0| + \sum_{n=1}^{\infty} \|p_n\|_{L^1[0,\infty)} \\
&= p_0 + \frac{\lambda\eta_1 p_0 \int_0^\infty e^{-\int_0^x \mu(\tau)d\tau} dx}{\eta_2 - \lambda\eta_1 \int_0^\infty e^{-\int_0^x \mu(\tau)d\tau} dx} \\
&= \frac{\eta_2 p_0}{\eta_2 - \lambda\eta_1 \int_0^\infty e^{-\int_0^x \mu(\tau)d\tau} dx} < \infty. \qquad (4\text{-}173)
\end{aligned}$$

In (4-173) we have used the following equality:

$$\begin{aligned}
&\eta_2 \int_0^\infty \mu(x) e^{-\int_0^x \mu(\tau)d\tau} dx - (q_2+\eta_2)\lambda\eta_1 \int_0^\infty x de^{-\int_0^x \mu(\tau)d\tau} \\
&= -\eta_2 e^{-\int_0^x \mu(\tau)d\tau}\Big|_{x=0}^{x=\infty} + \lambda\eta_1 x e^{-\int_0^x \mu(\tau)d\tau}\Big|_{x=0}^{x=\infty} \\
&\quad - \lambda\eta_1 \int_0^\infty e^{-\int_0^x \mu(\tau)d\tau} dx \\
&= \eta_2 - \lambda\eta_1 \int_0^\infty e^{-\int_0^x \mu(\tau)d\tau} dx.
\end{aligned}$$

(4-173) shows that 0 is an eigenvalue of $A + U + E$ (see Definition 1.22) and (4-168)–(4-170) mean that geometric multiplicity of 0 is 1 (see Definition 1.23). □

If we introduce $(A_0, D(A_0))$ by

$$\begin{aligned}
D(A_0) &:= \{p \in D(A_m) \mid Lp = 0\}, \\
A_0 p &:= A_m p,
\end{aligned}$$

then for any given $y \in X$, by considering the equation $(\gamma I - A_0)p = y$, that is,

$$(\gamma + \lambda\eta_1)p_0 = y_0 + \eta_2 \int_0^\infty p_1(x)\mu(x)dx, \tag{4-174}$$

$$\frac{dp_1(x)}{dx} = -(\gamma + \lambda\eta_1 + \mu(x))p_1(x) + y_1(x), \tag{4-175}$$

$$\begin{aligned}\frac{dp_n(x)}{dx} &= -(\gamma + \lambda\eta_1 + \mu(x))p_n(x) + \lambda\eta_1 p_{n-1}(x)\\ &\quad + y_n(x), \forall n \geq 2,\end{aligned} \tag{4-176}$$

$$p_n(0) = 0, \quad n \geq 1, \tag{4-177}$$

it is easy to calculate

$$\begin{aligned}p_0 = &\frac{1}{\gamma + \lambda\eta_1} y_0 + \frac{\eta_2}{\gamma + \lambda\eta_1} \int_0^\infty \mu(x) e^{-(\gamma+\lambda\eta_1)x - \int_0^x \mu(\tau)d\tau}\\ &\times \int_0^x y_1(\tau) e^{(\gamma+\lambda\eta_1)\tau + \int_0^\tau \mu(\xi)d\xi} d\tau dx,\end{aligned} \tag{4-178}$$

$$\begin{aligned}p_1(x) = &e^{-(\gamma+\lambda\eta_1)x - \int_0^x \mu(\tau)d\tau}\\ &\times \int_0^x y_1(\tau) e^{(\gamma+\lambda\eta_1)\tau + \int_0^\tau \mu(\xi)d\xi} d\tau,\end{aligned} \tag{4-179}$$

$$\begin{aligned}p_n(x) = &e^{-(\gamma+\lambda\eta_1)x - \int_0^x \mu(\tau)d\tau} \int_0^x y_n(\tau) e^{(\gamma+\lambda\eta_1)\tau + \int_0^\tau \mu(\xi)d\xi} d\tau\\ &+ \lambda\eta_1 e^{-(\gamma+\lambda\eta_1)x - \int_0^x \mu(\tau)d\tau}\\ &\times \int_0^x p_{n-1}(\tau) e^{(\gamma+\lambda\eta_1)\tau + \int_0^\tau \mu(\xi)d\xi} d\tau, \quad n \geq 2.\end{aligned} \tag{4-180}$$

If we set, $\forall f \in L^1[0, \infty)$,

$$Ef(x) := e^{-(\gamma+\lambda\eta_1)x - \int_0^x \mu(\tau)d\tau} \int_0^x f(\tau) e^{(\gamma+\lambda\eta_1)\tau + \int_0^\tau \mu(\xi)d\xi} d\tau,$$

then the above equations (4-178)–(4-180) give

$$(\gamma I - A_0)^{-1} \begin{pmatrix} y_0 \\ y_1(x) \\ y_2(x) \\ y_3(x) \\ y_4(x) \\ \vdots \end{pmatrix} = \begin{pmatrix} \frac{1}{\gamma+\lambda\eta_1} & \frac{\eta_2}{\gamma+\lambda\eta_1}\varphi E & 0 & 0 & 0 & \cdots \\ 0 & E & 0 & 0 & 0 & \cdots \\ 0 & \lambda\eta_1 E^2 & E & 0 & 0 & \cdots \\ 0 & (\lambda\eta_1)^2 E^3 & \lambda\eta_1 E^2 & E & 0 & \cdots \\ 0 & (\lambda\eta_1)^3 E^4 & (\lambda\eta_1)^2 E^3 & \lambda\eta_1 E^2 & E & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} y_0 \\ y_1(x) \\ y_2(x) \\ y_3(x) \\ y_4(x) \\ \vdots \end{pmatrix}.$$

From this together with the definition of resolvent set (see Definition 1.22) we have the following result.

Lemma 4.13. *Let* $\mu(x):[0,\infty)\to[0,\infty)$ *be measurable and* $0<\inf\limits_{x\in[0,\infty)}\mu(x)\le\sup\limits_{x\in[0,\infty)}\mu(x)<\infty$, *then*

$$\left\{\gamma\in\mathbb{C}\,\middle|\,\begin{array}{l}\Re\gamma+\lambda\eta_1>0,\\ \Re\gamma+\inf\limits_{x\in[0,\infty)}\mu(x)>0\end{array}\right\}\subset\rho(A_0),$$

which shows that all points on the imaginary axis belong to the resolvent set of A_0.

Proof. For any $f\in C_0^\infty[0,\infty)\cap L^1[0,\infty)$, by using integration by parts we estimate

$$\begin{aligned}
&\|Ef\|_{L^1[0,\infty)}=\int_0^\infty|Ef(x)|dx\\
&=\int_0^\infty\left|e^{-(\gamma+\lambda\eta_1)x-\int_0^x\mu(\xi)d\xi}\int_0^x e^{(\gamma+\lambda\eta_1)\tau+\int_0^\tau\mu(\xi)d\xi}f(\tau)d\tau\right|dx\\
&\le\int_0^\infty e^{-(\Re\gamma+\lambda\eta_1)x-\int_0^x\mu(\xi)d\xi}\int_0^x e^{(\Re\gamma+\lambda\eta_1)\tau+\int_0^\tau\mu(\xi)d\xi}|f(\tau)|d\tau dx\\
&=\int_0^\infty-\frac{1}{\Re\gamma+\lambda\eta_1+\mu(x)}\\
&\quad\times\int_0^x e^{(\Re\gamma+\lambda\eta_1)\tau+\int_0^\tau\mu(\xi)d\xi}|f(\tau)|d\tau de^{-(\Re\gamma+\lambda\eta_1)x-\int_0^x\mu(\xi)d\xi}\\
&\le-\frac{1}{\Re\gamma+\lambda\eta_1+\inf\limits_{x\in[0,\infty)}\mu(x)}\\
&\quad\times\int_0^\infty\int_0^x e^{(\Re\gamma+\lambda\eta_1)\tau+\int_0^\tau\mu(\xi)d\xi}|f(\tau)|d\tau de^{-(\Re\gamma+\lambda\eta_1)x-\int_0^x\mu(\xi)d\xi}\\
&=-\frac{1}{\Re\gamma+\lambda\eta_1+\inf\limits_{x\in[0,\infty)}\mu(x)}\left\{e^{-(\Re\gamma+\lambda\eta_1)x-\int_0^x\mu(\xi)d\xi}\right.\\
&\quad\times\int_0^x e^{(\Re\gamma+\lambda\eta_1)\tau+\int_0^\tau\mu(\tau)d\tau}|f(\tau)|d\tau\Bigg|_{x=0}^{x=\infty}\\
&\quad-\int_0^\infty e^{-(\Re\gamma+\lambda\eta_1)x-\int_0^x\mu(\tau)d\tau}\\
&\quad\left.\times e^{(\Re\gamma+\lambda\eta_1)x+\int_0^x\mu(\tau)d\tau}|f(x)|dx\right\}\\
&=-\frac{1}{\Re\gamma+\lambda\eta_1+\inf\limits_{x\in[0,\infty)}\mu(x)}\left\{\lim_{x\to\infty}e^{-(\Re\gamma+\lambda\eta_1)x-\int_0^x\mu(\xi)d\xi}\right.\\
&\quad\times\int_0^x e^{(\Re\gamma+\lambda\eta_1)\tau+\int_0^\tau\mu(\xi)d\xi}|f(\tau)|d\tau\\
&\quad\left.-\int_0^\infty|f(x)|dx\right\}
\end{aligned}$$

$$= \frac{1}{\Re\gamma + \lambda\eta_1 + \inf\limits_{x\in[0,\infty)} \mu(x)} \|f\|_{L^1[0,\infty)}. \tag{4-181}$$

Since $C_0^\infty[0,\infty)$ is dense in $L^1[0,\infty)$ by Adams [2], (4-181) holds for all $f \in L^1[0,\infty)$. (4-181) together with the condition $\Re\gamma + \inf\limits_{x\in[0,\infty)} \mu(x) > 0$ and $\|\varphi\| \leq \sup\limits_{x\in[0,\infty)} \mu(x)$ give, for $y \in X$,

$$\begin{aligned}
&\left\|(\gamma I - A_0)^{-1} y\right\| \\
&= \Bigg\| \Bigg(\frac{1}{\gamma + \lambda\eta_1} y_0 + \frac{\eta_2}{\gamma + \lambda\eta_1} \varphi E y_1, E y_1, \lambda\eta_1 E^2 y_1 + E y_2, \\
&\quad (\lambda\eta_1)^2 E^3 y_1 + \lambda\eta_1 E^2 y_2 + E y_3, \\
&\quad (\lambda\eta_1)^3 E^4 y_1 + (\lambda\eta_1)^2 E^3 y_2 + \lambda\eta_1 E^2 y_3 + E y_4, \ldots \Bigg) \Bigg\| \\
&= \left| \frac{1}{\gamma + \lambda\eta_1} y_0 + \frac{\eta_2}{\gamma + \lambda\eta_1} \varphi E y_1 \right| + \|E y_1\|_{L^1[0,\infty)} \\
&\quad + \left\|\lambda\eta_1 E^2 y_1 + E y_2\right\|_{L^1[0,\infty)} \\
&\quad + \left\|(\lambda\eta_1)^2 E^3 y_1 + \lambda\eta_1 E^2 y_2 + E y_3\right\|_{L^1[0,\infty)} \\
&\quad + \left\|(\lambda\eta_1)^3 E^4 y_1 + (\lambda\eta_1)^2 E^3 y_2 + \lambda\eta_1 E^2 y_3 + E y_4\right\|_{L^1[0,\infty)} \\
&\quad + \cdots \\
&\leq \left| \frac{1}{\gamma + \lambda\eta_1} \right| |y_0| + \frac{\eta_2}{|\gamma + \lambda\eta_1|} |\varphi E y_1| + \|E y_1\|_{L^1[0,\infty)} \\
&\quad + \left\|\lambda\eta_1 E^2 y_1\right\|_{L^1[0,\infty)} + \|E y_2\|_{L^1[0,\infty)} + \left\|(\lambda\eta_1)^2 E^3 y_1\right\|_{L^1[0,\infty)} \\
&\quad + \left\|\lambda\eta_1 E^2 y_2\right\|_{L^1[0,\infty)} + \|E y_3\|_{L^1[0,\infty)} + \left\|(\lambda\eta_1)^3 E^4 y_1\right\|_{L^1[0,\infty)} \\
&\quad + \left\|(\lambda\eta_1)^2 E^3 y_2\right\|_{L^1[0,\infty)} + \left\|\lambda\eta_1 E^2 y_3\right\|_{L^1[0,\infty)} + \|E y_4\|_{L^1[0,\infty)} \\
&\quad + \cdots \\
&\leq \frac{1}{|\gamma + \lambda\eta_1|} |y_0| + \frac{\eta_2}{|\gamma + \lambda\eta_1|} \|\varphi\| \|E\| \|y_1\|_{L^1[0,\infty)} \\
&\quad + \|E\| \|y_1\|_{L^1[0,\infty)} + \lambda\eta_1 \|E\|^2 \|y_1\|_{L^1[0,\infty)} \\
&\quad + \|E\| \|y_2\|_{L^1[0,\infty)} + (\lambda\eta_1)^2 \|E\|^3 \|y_1\|_{L^1[0,\infty)} \\
&\quad + \lambda\eta_1 \|E\|^2 \|y_2\|_{L^1[0,\infty)} + \|E\| \|y_3\|_{L^1[0,\infty)} \\
&\quad + (\lambda\eta_1)^3 \|E\|^4 \|y_1\|_{L^1[0,\infty)} + (\lambda\eta_1)^2 \|E\|^3 \|y_2\|_{L^1[0,\infty)} \\
&\quad + \lambda\eta_1 \|E\|^2 \|y_3\|_{L^1[0,\infty)} + \|E\| \|y_4\|_{L^1[0,\infty)} \\
&\quad + \cdots \\
&\leq \frac{1}{|\gamma + \lambda\eta_1|} |y_0| + \frac{\eta_2}{|\gamma + \lambda\eta_1|} \|\varphi\| \|E\| \|y_1\|_{L^1[0,\infty)}
\end{aligned}$$

$$
\begin{aligned}
&+\sum_{n=1}^{\infty}(\lambda\eta_1)^{n-1}\|E\|^n\|y_1\|_{L^1[0,\infty)}\\
&+\sum_{n=1}^{\infty}(\lambda\eta_1)^{n-1}\|E\|^n\|y_2\|_{L^1[0,\infty)}\\
&+\sum_{n=1}^{\infty}(\lambda\eta_1)^{n-1}\|E\|^n\|y_3\|_{L^1[0,\infty)}+\cdots\\
=&\frac{1}{|\gamma+\lambda\eta_1|}|y_0|+\frac{\eta_2}{|\gamma+\lambda\eta_1|}\|\varphi\|\|E\|\|y_1\|_{L^1[0,\infty)}\\
&+\sum_{n=1}^{\infty}(\lambda\eta_1)^{n-1}\|E\|^n\sum_{n=1}^{\infty}\|y_n\|_{L^1[0,\infty)}\\
\le&\frac{1}{|\gamma+\lambda\eta_1|}|y_0|+\frac{\eta_2}{|\gamma+\lambda\eta_1|}\frac{\sup\limits_{x\in[0,\infty)}\mu(x)}{\Re\gamma+\lambda\eta_1+\inf\limits_{x\in[0,\infty)}\mu(x)}\|y_1\|_{L^1[0,\infty)}\\
&+\sum_{n=1}^{\infty}(\lambda\eta_1)^{n-1}\left(\frac{1}{\Re\gamma+\lambda\eta_1+\inf\limits_{x\in[0,\infty)}\mu(x)}\right)^n\sum_{n=1}^{\infty}\|y_n\|_{L^1[0,\infty)}\\
=&\frac{1}{|\gamma+\lambda\eta_1|}|y_0|+\frac{\eta_2}{|\gamma+\lambda\eta_1|}\frac{\sup\limits_{x\in[0,\infty)}\mu(x)}{\Re\gamma+\lambda\eta_1+\inf\limits_{x\in[0,\infty)}\mu(x)}\|y_1\|_{L^1[0,\infty)}\\
&+\frac{1}{\Re\gamma+\inf\limits_{x\in[0,\infty)}\mu(x)}\sum_{n=1}^{\infty}\|y_n\|_{L^1[0,\infty)}\\
=&\sup\left\{\frac{1}{|\gamma+\lambda\eta_1|},\frac{\eta_2\sup\limits_{x\in[0,\infty)}\mu(x)}{|\gamma+\lambda\eta_1|(\Re\gamma+\lambda\eta_1+\inf\limits_{x\in[0,\infty)}\mu(x))}\right.\\
&\left.+\frac{1}{\Re\gamma+\inf\limits_{x\in[0,\infty)}\mu(x)}\right\}\|y\|<\infty.
\end{aligned}
$$

This shows that the result of this lemma is right. □

Lemma 4.14. *Let* $\mu(x):[0,\infty)\to[0,\infty)$ *be measurable and* $0<\inf\limits_{x\in[0,\infty)}\mu(x)\le\sup\limits_{x\in[0,\infty)}\mu(x)<\infty$, *then for* $\gamma\in\rho(A_0)$,

$$
\begin{aligned}
&p\in\ker(\gamma I-A_m)\\
&\Longleftrightarrow\\
&p(x)=(p_0,p_1(x),p_2(x),p_3(x),\dots),
\end{aligned}
$$

$$p_0 = \frac{\eta_2 a_1}{\gamma + \lambda\eta_1} \int_0^\infty \mu(x) e^{-(\gamma+\lambda\eta_1)x - \int_0^x \mu(\xi)d\xi} dx, \tag{4-182}$$

$$p_n(x) = e^{-(\gamma+\lambda\eta_1)x - \int_0^x \mu(\xi)d\xi} \sum_{k=0}^n \frac{(\lambda\eta_1 x)^k}{k!} a_{n-k}, \tag{4-183}$$

$$n \geq 1 \ and \ (a_1, a_2, a_3, \dots) \in l^1.$$

Proof. If $p \in \ker(\gamma I - A_m)$, then from the definition of A_m it is equivalent to the following equations:

$$(\gamma + \lambda\eta_1)p_0 = \eta_2 \int_0^\infty \mu(x)p_1(x)dx, \tag{4-184}$$

$$\frac{dp_1(x)}{dx} = -(\gamma + \lambda\eta_1 + \mu(x))p_1(x), \tag{4-185}$$

$$\frac{dp_n(x)}{dx} = -(\gamma + \lambda\eta_1 + \mu(x))p_n(x) + \lambda\eta_1 p_{n-1}(x), \ n \geq 2. \tag{4-186}$$

By solving (4-185) and (4-186) we have

$$p_1(x) = a_1 e^{-(\gamma+\lambda\eta_1)x - \int_0^x \mu(\tau)d\tau}, \tag{4-187}$$

$$\begin{aligned} p_n(x0 = {} & a_n e^{-(\gamma+\lambda\eta_1)x - \int_0^x \mu(\tau)d\tau} \\ & + \lambda\eta_1 e^{-(\gamma+\lambda\eta_1)x - \int_0^x \mu(\tau)d\tau} \\ & \times \int_0^x e^{(\gamma+\lambda\eta_1)\tau + \int_0^\tau \mu(\xi)d\xi} p_{n-1}(\tau)d\tau, \quad n \geq 2. \end{aligned} \tag{4-188}$$

Through inserting (4-187) into (4-184) it follows that

$$p_0 = \frac{a_1\eta_2}{\gamma + \lambda\eta_1} \int_0^\infty \mu(x) e^{-(\gamma+\lambda\eta_1)x - \int_0^x \mu(\tau)d\tau} dx. \tag{4-189}$$

By using (4-187) and (4-188) repeatedly we calculate

$$\begin{aligned} p_2(x) = {} & a_2 e^{-(\gamma+\lambda\eta_1)x - \int_0^x \mu(\tau)d\tau} \\ & + \lambda\eta_1 e^{-(\gamma+\lambda\eta_1)x - \int_0^x \mu(\tau)d\tau} \\ & \times \int_0^x e^{(\gamma+\lambda\eta_1)\tau + \int_0^\tau \mu(\tau)d\tau} p_1(\tau)d\tau \\ = {} & a_2 e^{-(\gamma+\lambda\eta_1)x - \int_0^x \mu(\tau)d\tau} \\ & + \lambda\eta_1 e^{-(\gamma+\lambda\eta_1)x - \int_0^x \mu(\tau)d\tau} \\ & \times \int_0^x e^{(\gamma+\lambda\eta_1)\tau + \int_0^\tau \mu(\xi)d\xi} a_1 e^{-(\gamma+\lambda\eta_1)\tau - \int_0^\tau \mu(\xi)d\xi} d\tau \\ = {} & e^{-(\gamma+\lambda\eta_1)x - \int_0^x \mu(\tau)d\tau}[a_2 + a_1\lambda\eta_1 x], \end{aligned} \tag{4-190}$$

$$\begin{aligned}
p_3(x) &= a_3 e^{-(\gamma+\lambda\eta_1)x-\int_0^x \mu(\tau)d\tau} \\
&\quad + \lambda\eta_1 e^{-(\gamma+\lambda\eta_1)x-\int_0^x \mu(\tau)d\tau} \\
&\quad \times \int_0^x e^{(\gamma+\lambda\eta_1)\tau+\int_0^\tau \mu(\xi)d\xi} p_2(\tau)d\tau \\
&= a_3 e^{-(\gamma+\lambda\eta_1)x-\int_0^x \mu(\tau)d\tau} \\
&\quad + \lambda\eta_1 e^{-(\gamma+\lambda\eta_1)x-\int_0^x \mu(\tau)d\tau} \int_0^x [a_2 + a_1\lambda\eta_1\tau]d\tau \\
&= e^{-(\gamma+\lambda\eta_1)x-\int_0^x \mu(\tau)d\tau} \left[a_3 + a_2\lambda\eta_1 x + a_1 \frac{(\lambda\eta_1 x)^2}{2!}\right] \\
&= e^{-(\gamma+\lambda\eta_1)x-\int_0^x \mu(\tau)d\tau} \sum_{k=0}^{2} a_{3-k} \frac{(\lambda\eta_1 x)^k}{k!}, \qquad (4\text{-}191)
\end{aligned}$$

$$\begin{aligned}
p_4(x) &= a_4 e^{-(\gamma+\lambda\eta_1)x-\int_0^x \mu(\tau)d\tau} \\
&\quad + \lambda\eta_1 e^{-(\gamma+\lambda\eta_1)x-\int_0^x \mu(\tau)d\tau} \\
&\quad \times \int_0^x \left[a_3 + a_2\lambda\eta_1\tau + a_1 \frac{(\lambda\eta_1\tau)^2}{2!}\right] d\tau \\
&= e^{-(\gamma+\lambda\eta_1)x-\int_0^x \mu(\tau)d\tau} \sum_{k=0}^{3} a_{4-k} \frac{(\lambda\eta_1 x)^k}{k!}, \qquad (4\text{-}192)
\end{aligned}$$

$$\cdots\cdots$$

$$p_n(x) = a_n e^{-(\gamma+\lambda\eta_1)x-\int_0^x \mu(\tau)d\tau} \sum_{k=0}^{n-1} a_{n-k} \frac{(\lambda\eta_1 x)^k}{k!}, \qquad (4\text{-}193)$$

$$\cdots\cdots$$

Since $p \in \ker(\gamma I - A_m)$, $p \in D(A_m)$ implies by the imbedding theorem (see Adams [2])

$$\begin{aligned}
\sum_{n=1}^{\infty} |a_n| &= \sum_{n=1}^{\infty} |p_n(0)| \leq \sum_{n=1}^{\infty} \|p_n\|_{L^\infty[0,\infty)} \\
&\leq \sum_{n=1}^{\infty} \|p_n\|_{L^1[0,\infty)} + \sum_{n=1}^{\infty} \left\| \frac{dp_n(x)}{dx} \right\|_{L^1[0,\infty)} < \infty,
\end{aligned}$$

from which together with (4-189)–(4-193) we know that (4-182)–(4-183) hold.

Conversely, if (4-182)–(4-183) are true, then by using $\int_0^\infty e^{-bx} x^k dx = \frac{k!}{b^{k+1}}$, $k \geq 1$, $b > 0$, integration by parts and the Fubini theorem (see Theorem 1.49) we estimate

$$\|p_n\|_{L^1[[0,\infty)} = \int_0^\infty \left| e^{-(\gamma+\lambda\eta_1)x-\int_0^x \mu(\tau)d\tau} \sum_{k=0}^{n-1} a_{n-k} \frac{(\lambda\eta_1 x)^k}{k!} \right| dx$$

$$\leq \int_0^\infty e^{-(\Re\gamma+\lambda\eta_1)x-\int_0^x \mu(\tau)d\tau} \sum_{k=0}^{n-1} |a_{n-k}| \frac{(\lambda\eta_1 x)^k}{k!} dx$$
$$= \sum_{k=0}^{n-1} |a_{n-k}| \frac{(\lambda\eta_1)^k}{k!} \int_0^\infty e^{-(\Re\gamma+\lambda\eta_1)x-\int_0^x \mu(\tau)d\tau} x^k dx$$
$$\leq \sum_{k=0}^{n-1} |a_{n-k}| \frac{(\lambda\eta_1)^k}{k!} \int_0^\infty e^{-\left(\Re\gamma+\lambda\eta_1+\inf_{x\in[0,\infty)}\mu(x)\right)x} x^k dx$$
$$= \sum_{k=0}^{n-1} |a_{n-k}| \frac{(\lambda\eta_1)^k}{k!} \frac{k!}{\left(\Re\gamma+\lambda\eta_1+\inf\limits_{x\in[0,\infty)}\mu(x)\right)^{k+1}}$$
$$= \sum_{k=0}^{n-1} \frac{(\lambda\eta_1)^k}{\left(\Re\gamma+\lambda\eta_1+\inf\limits_{x\in[0,\infty)}\mu(x)\right)^{k+1}} |a_{n-k}|$$
$$\Longrightarrow$$
$$|p_0| + \sum_{n=1}^\infty \|p_n\|_{L^1[0,\infty)}$$
$$\leq |a_1| \frac{\eta_2}{|\gamma+\lambda\eta_1|} \int_0^\infty \mu(x) e^{-(\Re\gamma+\lambda\eta_1)x-\int_0^x \mu(\tau)d\tau} dx$$
$$+ \sum_{n=1}^\infty \sum_{k=0}^{n-1} \frac{(\lambda\eta_1)^k}{\left(\Re\gamma+\lambda\eta_1+\inf\limits_{x\in[0,\infty)}\mu(x)\right)^{k+1}} |a_{n-k}|$$
$$= |a_1| \frac{\eta_2}{|\gamma+\lambda\eta_1|} \int_0^\infty \mu(x) e^{-(\Re\gamma+\lambda\eta_1)x-\int_0^x \mu(\tau)d\tau} dx$$
$$+ \sum_{k=0}^\infty \sum_{n=k+1}^\infty \frac{(\lambda\eta_1)^k}{\left(\Re\gamma+\lambda\eta_1+\inf\limits_{x\in[0,\infty)}\mu(x)\right)^{k+1}} |a_{n-k}|$$
$$\leq |a_1| \frac{\eta_2}{|\gamma+\lambda\eta_1|} \int_0^\infty \mu(x) e^{-\int_0^x \mu(\tau)d\tau} dx$$
$$+ \sum_{k=0}^\infty \frac{(\lambda\eta_1)^k}{\left(\Re\gamma+\lambda\eta_1+\inf\limits_{x\in[0,\infty)}\mu(x)\right)^{k+1}} \sum_{n=k+1}^\infty |a_{n-k}|$$
$$\leq |a_1| \frac{\eta_2}{|\gamma+\lambda\eta_1|}$$
$$+ \frac{1}{\Re\gamma+\lambda\eta_1+\inf\limits_{x\in[0,\infty)}\mu(x)}$$

$$\times \sum_{k=0}^{\infty} \left(\frac{\lambda\eta_1}{\Re\gamma + \lambda\eta_1 + \inf_{x\in[0,\infty)} \mu(x)} \right)^k \sum_{n=1}^{\infty} |a_n|$$

$$= \frac{\eta_2}{|\gamma + \lambda\eta_1|}|a_1|$$

$$+ \frac{1}{\Re\gamma + \inf_{x\in[0,\infty)} \mu(x)} \sum_{n=1}^{\infty} |a_n| < \infty. \tag{4-194}$$

From this together with

$$\frac{dp_1(x)}{dx} = -(\gamma + \lambda\eta_1 + \mu(x))p_1(x),$$
$$\frac{dp_n(x)}{dx} = -(\gamma + \lambda\eta_1 + \mu(x))p_n(x) + \lambda\eta_1 p_{n-1}(x), \quad n \geq 2,$$

we immediately get

$$\sum_{n=1}^{\infty} \left\| \frac{dp_n}{dx} \right\|_{L^1[0,\infty)} = \sum_{n=1}^{\infty} \int_0^{\infty} \left| \frac{dp_n(x)}{dx} \right| dx$$

$$\leq \sum_{n=1}^{\infty} \int_0^{\infty} |\gamma + \lambda\eta_1 + \mu(x)||p_n(x)|dx$$

$$+ \sum_{n=2}^{\infty} \int_0^{\infty} \lambda\eta_1 |p_{n-1}(x)|dx$$

$$\leq \sum_{n=1}^{\infty} (|\gamma| + \lambda\eta_1 + \sup_{x\in[0,\infty)} \mu(x)) \|p_n\|_{L^1[0,\infty)}$$

$$+ \lambda\eta_1 \sum_{n=2}^{\infty} \|p_{n-1}\|_{L^1[0,\infty)}$$

$$= \left(|\gamma| + 2\lambda\eta_1 + \sup_{x\in[0,\infty)} \mu(x) \right) \sum_{n=1}^{\infty} \|p_n\|_{L^1[0,\infty)}$$

$$\leq \frac{|\gamma| + 2\lambda\eta_1 + \sup_{x\in[0,\infty)} \mu(x)}{\Re\gamma + \inf_{x\in[0,\infty)} \mu(x)} \sum_{n=1}^{\infty} |a_n|$$

$$< \infty. \tag{4-195}$$

The above formulas show $p \in \ker(\gamma I - A_m)$. $\square$

Observe that the operator L is surjective. Hence,

$$L|_{\ker(\gamma I - A_m)} : \ker(\gamma I - A_m) \longrightarrow \partial X$$

is invertible if $\gamma \in \rho(A_0)$. Thus we introduce the Dirichlet operator D_γ as

$$D_\gamma := \left(L\left|_{\ker(\gamma I - A_m)}\right.\right)^{-1} : \partial X \longrightarrow \ker(\gamma I - A_m). \tag{4-196}$$

Lemma 4.14 gives the explicit form of D_γ for $\gamma \in \rho(A_0)$,

$$D_\gamma \begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ \vdots \end{pmatrix} = \begin{pmatrix} \mho_{11} & 0 & 0 & 0 & \cdots \\ \mho_{21} & 0 & 0 & 0 & \cdots \\ \mho_{31} & \mho_{32} & 0 & 0 & \cdots \\ \mho_{41} & \mho_{42} & \mho_{43} & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ \vdots \end{pmatrix}, \tag{4-197}$$

here

$$\begin{aligned}
\mho_{11} &= \frac{\eta_2}{\gamma + \lambda\eta_1} \int_0^\infty \mu(x) e^{-(\gamma+\lambda\eta_1)x - \int_0^x \mu(\tau)d\tau} dx, \\
\mho_{21} &= e^{-(\gamma+\lambda\eta_1)x - \int_0^x \mu(\tau)d\tau}, \\
\mho_{31} &= \lambda\eta_1 x e^{-(\gamma+\lambda\eta_1)x - \int_0^x \mu(\tau)d\tau}, \\
\mho_{32} &= e^{-(\gamma+\lambda\eta_1)x - \int_0^x \mu(\tau)d\tau}, \\
\mho_{41} &= \frac{(\lambda\eta_1 x)^2}{2!} e^{-(\gamma+\lambda\eta_1)x - \int_0^x \mu(\tau)d\tau}, \\
\mho_{42} &= \lambda\eta_1 x e^{-(\gamma+\lambda\eta_1)x - \int_0^x \mu(\tau)d\tau}, \\
\mho_{43} &= e^{-(\gamma+\lambda\eta_1)x - \int_0^x \mu(\tau)d\tau}, \\
&\cdots\cdots \\
\mho_{ij} &= \frac{(\lambda\eta_1 x)^{i-j-1}}{(i-j-1)!} e^{-(\gamma+\lambda\eta_1)x - \int_0^x \mu(\tau)d\tau}, \\
&\quad i = 3, 4, 5, \ldots; j = 1, 2, \ldots, i-1, \\
&\cdots\cdots
\end{aligned}$$

From this together with the definition of Φ we have

$$\Phi D_\gamma \begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ \vdots \end{pmatrix} = \begin{pmatrix} \kappa_{11} & \kappa_{12} & 0 & 0 & 0 & \cdots \\ \kappa_{21} & \kappa_{22} & \kappa_{23} & 0 & 0 & \cdots \\ \kappa_{31} & \kappa_{32} & \kappa_{33} & \kappa_{34} & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ \vdots \end{pmatrix}, \tag{4-198}$$

where

$$\begin{aligned}
\kappa_{11} &= \left(\frac{\lambda\eta_1\eta_2}{\gamma+\lambda\eta_1}+q_2\right)\int_0^\infty \mu(x)e^{-(\gamma+\lambda\eta_1)x-\int_0^x \mu(\tau)d\tau}dx \\
&\quad + \eta_2\int_0^\infty \lambda\eta_1 x\mu(x)e^{-(\gamma+\lambda\eta_1)x-\int_0^x \mu(\tau)d\tau}dx, \\
\kappa_{12} &= \eta_2\int_0^\infty \mu(x)e^{-(\gamma+\lambda\eta_1)x-\int_0^x \mu(\tau)d\tau}dx, \\
\kappa_{21} &= \eta_2\int_0^\infty \frac{(\lambda\eta_1 x)^2}{2!}\mu(x)e^{-(\gamma+\lambda\eta_1)x-\int_0^x \mu(\tau)d\tau}dx \\
&\quad + q_2\int_0^\infty \lambda\eta_1 x\mu(x)e^{-(\gamma+\lambda\eta_1)x-\int_0^x \mu(\tau)d\tau}dx, \\
\kappa_{22} &= \eta_2\int_0^\infty \lambda\eta_1 x\mu(x)e^{-(\gamma+\lambda\eta_1)x-\int_0^x \mu(\tau)d\tau}dx \\
&\quad + q_2\int_0^\infty \mu(x)e^{-(\gamma+\lambda\eta_1)x-\int_0^x \mu(\tau)d\tau}dx, \\
\kappa_{23} &= \eta_2\int_0^\infty \mu(x)e^{-(\gamma+\lambda\eta_1)x-\int_0^x \mu(\tau)d\tau}dx, \\
\kappa_{31} &= \eta_2\int_0^\infty \frac{(\lambda\eta_1 x)^3}{3!}\mu(x)e^{-(\gamma+\lambda\eta_1)x-\int_0^x \mu(\tau)d\tau}dx \\
&\quad + q_2\int_0^\infty \frac{(\lambda\eta_1 x)^2}{2!}\mu(x)e^{-(\gamma+\lambda\eta_1)x-\int_0^x \mu(\tau)d\tau}dx, \\
\kappa_{32} &= \eta_2\int_0^\infty \frac{(\lambda\eta_1 x)^2}{2!}\mu(x)e^{-(\gamma+\lambda\eta_1)x-\int_0^x \mu(\tau)d\tau}dx \\
&\quad + q_2\int_0^\infty \lambda\eta_1 x\mu(x)e^{-(\gamma+\lambda\eta_1)x-\int_0^x \mu(\tau)d\tau}dx, \\
\kappa_{33} &= \eta_2\int_0^\infty \lambda\eta_1 x\mu(x)e^{-(\gamma+\lambda\eta_1)x-\int_0^x \mu(\tau)d\tau}dx \\
&\quad + q_2\int_0^\infty \mu(x)e^{-(\gamma+\lambda\eta_1)x-\int_0^x \mu(\tau)d\tau}dx, \\
\kappa_{34} &= \eta_2\int_0^\infty \mu(x)e^{-(\gamma+\lambda\eta_1)x-\int_0^x \mu(\tau)d\tau}dx, \\
&\cdots\cdots \\
\kappa_{ij} &= \eta_2\int_0^\infty \frac{(\lambda\eta_1 x)^{i+1-j}}{(i+1-j)!}\mu(x)e^{-(\gamma+\lambda\eta_1)x-\int_0^x \mu(\tau)d\tau}dx \\
&\quad + q_2\int_0^\infty \frac{(\lambda\eta_1 x)^{i-j}}{(i-j)!}\mu(x)e^{-(\gamma+\lambda\eta_1)x-\int_0^x \mu(\tau)d\tau}dx, \\
&\quad i=2,3,4,\ldots;\ j=1,2,3,\ldots,i,
\end{aligned}$$

$$\kappa_{i,i+1} = \eta_2 \int_0^\infty \mu(x) e^{-(\gamma+\lambda\eta_1)x - \int_0^x \mu(\tau)d\tau} dx, \; i = 1, 2, 3, \dots,$$
$$\dots\dots$$

The following result can be found in Radl [97].

Lemma 4.15. *Assume* $\gamma \in \rho(A_0)$ *and there exists* $\gamma_0 \in \mathbb{C}$ *such that* $1 \notin \sigma(\Phi D_{\gamma_0})$. *Then*

$$\gamma \in \sigma(A + U + E) \Longleftrightarrow 1 \in \sigma(\Phi D_\gamma).$$

From this together with Theorem 1.88 we derive the following important result.

Lemma 4.16. *Let* $\mu(x) : [0, \infty) \to [0, \infty)$ *be measurable and* $0 < \inf\limits_{x\in[0,\infty)} \mu(x) \le \sup\limits_{x\in[0,\infty)} \mu(x) < \infty$, *then all points on the imaginary axis except zero belong to the resolvent set of* $A + U + E$.

Proof. Let $\gamma = ia$, $a \neq 0$ and $\quad = (a_1, a_2, a_3, \dots) \in l^1$. From the Riemann-Lebesgue lemma

$$\lim_{a\to\infty} \int_0^\infty f(x)\cos(ax)dx = 0, \; \lim_{a\to\infty} \int_0^\infty f(x)\sin(ax)dx = 0$$

we know that there exists a positive constant $\mathbb{M}$ such that for $|a| > \mathbb{M}$,

$$\begin{aligned}
&\left| \int_0^\infty \frac{(\lambda\eta_1 x)^k}{k!} \mu(x) e^{-(\gamma+\lambda\eta_1)x - \int_0^x \mu(\tau)d\tau} dx \right| \\
&= \left| \int_0^\infty \frac{(\lambda\eta_1 x)^k}{k!} \mu(x) e^{-\lambda\eta_1 x - \int_0^x \mu(\tau)d\tau} [\cos(ax) - i\sin(ax)] dx \right| \\
&= \left| \int_0^\infty \frac{(\lambda\eta_1 x)^k}{k!} \mu(x) e^{-\lambda\eta_1 x - \int_0^x \mu(\tau)d\tau} \cos(ax) dx \right. \\
&\quad \left. - i \int_0^\infty \frac{(\lambda\eta_1 x)^k}{k!} \mu(x) e^{-\lambda\eta_1 x - \int_0^x \mu(\tau)d\tau} \sin(ax) dx \right| \\
&= \left\{ \left(\int_0^\infty \frac{(\lambda\eta_1 x)^k}{k!} \mu(x) e^{-\lambda\eta_1 x - \int_0^x \mu(\tau)d\tau} \cos(ax) dx \right)^2 \right. \\
&\quad \left. + \left(\int_0^\infty \frac{(\lambda\eta_1 x)^k}{k!} \mu(x) e^{-\lambda\eta_1 x - \int_0^x \mu(\tau)d\tau} \sin(ax) dx \right)^2 \right\}^{\frac{1}{2}} \\
&< \int_0^\infty \frac{(\lambda\eta_1 x)^k}{k!} \mu(x) e^{-\lambda\eta_1 x - \int_0^x \mu(\tau)d\tau} dx, \; k \ge 0.
\end{aligned} \tag{4-199}$$

This together with $\frac{\lambda\eta_1}{|ia+\lambda\eta_1|} = \frac{\lambda\eta_1}{\sqrt{a^2+(\lambda\eta)^2}} < 1$ and $\eta_2 + q_2 = 1$ give

$$
\begin{aligned}
\|\Phi D_\gamma \ \| \leq & \left\{ \left| \left(\frac{\lambda\eta_1\eta_2}{\gamma+\lambda\eta_1} + q_2 \right) \int_0^\infty \mu(x) e^{-(\gamma+\lambda\eta_1)x - \int_0^x \mu(\tau)d\tau} dx \right. \right. \\
& \left. + \eta_2 \int_0^\infty \lambda\eta_1 x \mu(x) e^{-(\gamma+\lambda\eta_1)x - \int_0^x \mu(\tau)d\tau} dx \right| \\
& + \left| \eta_2 \int_0^\infty \frac{(\lambda\eta_1 x)^2}{2!} \mu(x) e^{-(\gamma+\lambda\eta_1)x - \int_0^x \mu(\tau)d\tau} dx \right. \\
& \left. + q_2 \int_0^\infty \lambda\eta_1 x \mu(x) e^{-(\gamma+\lambda\eta_1)x - \int_0^x \mu(\tau)d\tau} dx \right| \\
& + \left| \eta_2 \int_0^\infty \frac{(\lambda\eta_1 x)^3}{3!} \mu(x) e^{-(\gamma+\lambda\eta_1)x - \int_0^x \mu(\tau)d\tau} dx \right. \\
& \left. + q_2 \int_0^\infty \frac{(\lambda\eta_1 x)^2}{2!} \mu(x) e^{-(\gamma+\lambda\eta_1)x - \int_0^x \mu(\tau)d\tau} dx \right| \\
& + \left| \eta_2 \int_0^\infty \frac{(\lambda\eta_1 x)^4}{4!} \mu(x) e^{-(\gamma+\lambda\eta_1)x - \int_0^x \mu(\tau)d\tau} dx \right. \\
& \left. + q_2 \int_0^\infty \frac{(\lambda\eta_1 x)^3}{3!} \mu(x) e^{-(\gamma+\lambda\eta_1)x - \int_0^x \mu(\tau)d\tau} dx \right| \\
& \left. + \cdots\cdots \right\} |a_1| \\
& + \left\{ \left| \eta_2 \int_0^\infty \mu(x) e^{-(\gamma+\lambda\eta_1)x - \int_0^x \mu(\tau)d\tau} dx \right| \right. \\
& + \left| \eta_2 \int_0^\infty \lambda\eta_1 x \mu(x) e^{-(\gamma+\lambda\eta_1)x - \int_0^x \mu(\tau)d\tau} dx \right. \\
& \left. + q_2 \int_0^\infty \mu(x) e^{-(\gamma+\lambda\eta_1)x - \int_0^x \mu(\tau)d\tau} dx \right| \\
& + \left| \eta_2 \int_0^\infty \frac{(\lambda\eta_1 x)^2}{2!} \mu(x) e^{-(\gamma+\lambda\eta_1)x - \int_0^x \mu(\tau)d\tau} dx \right. \\
& \left. + q_2 \int_0^\infty \lambda\eta_1 x \mu(x) e^{-(\gamma+\lambda\eta_1)x - \int_0^x \mu(\tau)d\tau} dx \right| \\
& + \left| \eta_2 \int_0^\infty \frac{(\lambda\eta_1 x)^3}{3!} \mu(x) e^{-(\gamma+\lambda\eta_1)x - \int_0^x \mu(\tau)d\tau} dx \right. \\
& \left. + q_2 \int_0^\infty \frac{(\lambda\eta_1 x)^2}{2!} \mu(x) e^{-(\gamma+\lambda\eta_1)x - \int_0^x \mu(\tau)d\tau} dx \right| \\
& + \left| \eta_2 \int_0^\infty \frac{(\lambda\eta_1 x)^4}{4!} \mu(x) e^{-(\gamma+\lambda\eta_1)x - \int_0^x \mu(\tau)d\tau} dx \right. \\
& \left. + q_2 \int_0^\infty \frac{(\lambda\eta_1 x)^3}{3!} \mu(x) e^{-(\gamma+\lambda\eta_1)x - \int_0^x \mu(\tau)d\tau} dx \right|
\end{aligned}
$$

$$
\begin{aligned}
&+\cdots\cdots\Big\}(|a_2|+|a_3|+|a_4|+\cdots)\\
=&\left\{\left|\left(\frac{\lambda\eta_1\eta_2}{ia+\lambda\eta_1}+q_2\right)\int_0^\infty \mu(x)e^{-(ia+\lambda\eta_1)x-\int_0^x\mu(\tau)d\tau}dx\right.\right.\\
&\left.+\eta_2\int_0^\infty \lambda\eta_1 x\mu(x)e^{-(ia+\lambda\eta_1)x-\int_0^x\mu(\tau)d\tau}dx\right|\\
&+\left|\eta_2\int_0^\infty \frac{(\lambda\eta_1 x)^2}{2!}\mu(x)e^{-(ia+\lambda\eta_1)x-\int_0^x\mu(\tau)d\tau}dx\right.\\
&\left.+q_2\int_0^\infty \lambda\eta_1 x\mu(x)e^{-(ia+\lambda\eta_1)x-\int_0^x\mu(\tau)d\tau}dx\right|\\
&+\left|\eta_2\int_0^\infty \frac{(\lambda\eta_1 x)^3}{3!}\mu(x)e^{-(ia+\lambda\eta_1)x-\int_0^x\mu(\tau)d\tau}dx\right.\\
&\left.+q_2\int_0^\infty \frac{(\lambda\eta_1 x)^2}{2!}\mu(x)e^{-(ia+\lambda\eta_1)x-\int_0^x\mu(\tau)d\tau}dx\right|\\
&+\left|\eta_2\int_0^\infty \frac{(\lambda\eta_1 x)^4}{4!}\mu(x)e^{-(ia+\lambda\eta_1)x-\int_0^x\mu(\tau)d\tau}dx\right.\\
&\left.+q_2\int_0^\infty \frac{(\lambda\eta_1 x)^3}{3!}\mu(x)e^{-(ia+\lambda\eta_1)x-\int_0^x\mu(\tau)d\tau}dx\right|\\
&+\cdots\cdots\Big\}|a_1|\\
&+\left\{\left|\eta_2\int_0^\infty \mu(x)e^{-(ia+\lambda\eta_1)x-\int_0^x\mu(\tau)d\tau}dx\right|\right.\\
&+\left|\eta_2\int_0^\infty \lambda\eta_1 x\mu(x)e^{-(ia+\lambda\eta_1)x-\int_0^x\mu(\tau)d\tau}dx\right.\\
&\left.+q_2\int_0^\infty \mu(x)e^{-(ia+\lambda\eta_1)x-\int_0^x\mu(\tau)d\tau}dx\right|\\
&+\left|\eta_2\int_0^\infty \frac{(\lambda\eta_1 x)^2}{2!}\mu(x)e^{-(ia+\lambda\eta_1)x-\int_0^x\mu(\tau)d\tau}dx\right.\\
&\left.+q_2\int_0^\infty \lambda\eta_1 x\mu(x)e^{-(ia+\lambda\eta_1)x-\int_0^x\mu(\tau)d\tau}dx\right|\\
&+\left|\eta_2\int_0^\infty \frac{(\lambda\eta_1 x)^3}{3!}\mu(x)e^{-(ia+\lambda\eta_1)x-\int_0^x\mu(\tau)d\tau}dx\right.\\
&\left.+q_2\int_0^\infty \frac{(\lambda\eta_1 x)^2}{2!}\mu(x)e^{-(ia+\lambda\eta_1)x-\int_0^x\mu(\tau)d\tau}dx\right|\\
&+\left|\eta_2\int_0^\infty \frac{(\lambda\eta_1 x)^4}{4!}\mu(x)e^{-(ia+\lambda\eta_1)x-\int_0^x\mu(\tau)d\tau}dx\right.
\end{aligned}
$$

$$
\begin{aligned}
&+q_2 \int_0^\infty \frac{(\lambda\eta_1 x)^3}{3!}\mu(x)e^{-(ia+\lambda\eta_1)x-\int_0^x \mu(\tau)d\tau}dx\Bigg| \\
&+\cdots\cdots\Bigg\}(|a_2|+|a_3|+|a_4|+\cdots) \\
<&\left\{\frac{\lambda\eta_1\eta_2}{|ia+\lambda\eta_1|}\int_0^\infty \mu(x)e^{-\lambda\eta_1 x-\int_0^x \mu(\tau)d\tau}dx\right. \\
&+\eta_2\int_0^\infty \sum_{n=1}^\infty \frac{(\lambda\eta_1 x)^n}{n!}\mu(x)e^{-\lambda\eta_1 x-\int_0^x \mu(\tau)d\tau}dx \\
&\left.+q_2\int_0^\infty \sum_{n=0}^\infty \frac{(\lambda\eta_1 x)^n}{n!}\mu(x)e^{-\lambda\eta_1 x-\int_0^x \mu(\tau)d\tau}dx\right\}|a_1| \\
&+\left\{\eta_2\int_0^\infty \sum_{n=0}^\infty \frac{(\lambda\eta_1 x)^n}{n!}\mu(x)e^{-\lambda\eta_1 x-\int_0^x \mu(\tau)d\tau}dx\right. \\
&\left.+q_2\int_0^\infty \sum_{n=0}^\infty \frac{(\lambda\eta_1 x)^n}{n!}\mu(x)e^{-\lambda\eta_1 x-\int_0^x \mu(\tau)d\tau}dx\right\} \\
&\times(|a_2|+|a_3|+|a_4|+\cdots) \\
<&\left\{\eta_2\int_0^\infty \mu(x)e^{-\lambda\eta_1 x-\int_0^x \mu(\tau)d\tau}dx\right. \\
&+\eta_2\int_0^\infty \sum_{n=1}^\infty \frac{(\lambda\eta_1 x)^n}{n!}\mu(x)e^{-\lambda\eta_1 x-\int_0^x \mu(\tau)d\tau}dx \\
&\left.+q_2\int_0^\infty e^{\lambda\eta_1 x}\mu(x)e^{-\lambda\eta_1 x-\int_0^x \mu(\tau)d\tau}dx\right\}|a_1| \\
&+\left\{\eta_2\int_0^\infty e^{\lambda\eta_1 x}\mu(x)e^{-\lambda\eta_1 x-\int_0^x \mu(\tau)d\tau}dx\right. \\
&\left.+q_2\int_0^\infty e^{\lambda\eta_1 x}\mu(x)e^{-\lambda\eta_1 x-\int_0^x \mu(\tau)d\tau}dx\right\}\sum_{n=1}^\infty |a_n| \\
=&\left\{\eta_2\int_0^\infty e^{\lambda\eta_1 x}\mu(x)e^{-\lambda\eta_1 x-\int_0^x \mu(\tau)d\tau}dx\right. \\
&\left.+q_2\int_0^\infty \mu(x)e^{-\int_0^x \mu(\tau)d\tau}dx\right\}|a_1| \\
&+\left\{\eta_2\int_0^\infty \mu(x)e^{-\int_0^x \mu(\tau)d\tau}dx\right. \\
&\left.+q_2\int_0^x \mu(x)e^{-\int_0^x \mu(\tau)d\tau}dx\right\}\sum_{n=2}^\infty |a_n| \\
=&\left\{-\eta_2 e^{-\int_0^x \mu(\tau)d\tau}\Big|_{x=0}^{x=\infty} - q_2 e^{-\int_0^x \mu(\tau)d\tau}\Big|_{x=0}^{x=\infty}\right\}|a_1|
\end{aligned}
$$

$$+\left\{-\eta_2 e^{-\int_0^x \mu(\tau)d\tau}\Big|_{x=0}^{x=\infty} - q_2 e^{-\int_0^x \mu(\tau)d\tau}\Big|_{x=0}^{x=\infty}\right\}\sum_{n=2}^{\infty}|a_n|$$

$$= (\eta_2 + q_2)|a_1| + (\eta_2 + q_2)\sum_{n=2}^{\infty}|a_n|$$

$$= \sum_{n=1}^{\infty}|a_n| = \|\ \ \|$$

$$\Longrightarrow$$

$$\|\Phi D_\gamma\| < 1. \tag{4-200}$$

In the above formula it is impossible that $\|\Phi D_\gamma\| = 1$. In fact, it is evident that for all $k \geq 0$,

$$\left|\int_0^\infty \frac{(\lambda\eta_1 x)^k}{k!}\mu(x)e^{-(ia+\lambda\eta_1)x-\int_0^x \mu(\tau)d\tau}dx\right|$$
$$\leq \int_0^\infty \frac{(\lambda\eta_1 x)^k}{k!}\mu(x)\left|e^{-(ia+\lambda\eta_1)x-\int_0^x \mu(\tau)d\tau}\right|dx$$
$$= \int_0^\infty \frac{(\lambda\eta_1 x)^k}{k!}\mu(x)e^{-\lambda\eta_1 x-\int_0^x \mu(\tau)d\tau}dx.$$

By using (4-199) for finite $k \geq 0$, we know that the coefficient of $|a_i|$ $(i \geq 2)$ satisfies, for $|a| > \mathbb{M}$,

$$\left|\eta_2\int_0^\infty \mu(x)e^{-(ia+\lambda\eta_1)x-\int_0^x \mu(\tau)d\tau}dx\right|$$
$$+\left|\eta_2\int_0^\infty \lambda\eta_1 x\mu(x)e^{-(ia+\lambda\eta_1)x-\int_0^x \mu(\tau)d\tau}dx\right.$$
$$\left.+q_2\int_0^\infty \mu(x)e^{-(ia+\lambda\eta_1)x-\int_0^x \mu(\tau)d\tau}dx\right|$$
$$+\left|\eta_2\int_0^\infty \frac{(\lambda\eta_1 x)^2}{2!}\mu(x)e^{-(ia+\lambda\eta_1)x-\int_0^x \mu(\tau)d\tau}dx\right.$$
$$\left.+q_2\int_0^\infty \lambda\eta_1 x\mu(x)e^{-(ia+\lambda\eta_1)x-\int_0^x \mu(\tau)d\tau}dx\right|$$
$$+\left|\eta_2\int_0^\infty \frac{(\lambda\eta_1 x)^3}{3!}\mu(x)e^{-(ia+\lambda\eta_1)x-\int_0^x \mu(\tau)d\tau}dx\right.$$
$$\left.+q_2\int_0^\infty \frac{(\lambda\eta_1 x)^2}{2!}\mu(x)e^{-(ia+\lambda\eta_1)x-\int_0^x \mu(\tau)d\tau}dx\right|$$
$$+\left|\eta_2\int_0^\infty \frac{(\lambda\eta_1 x)^4}{4!}\mu(x)e^{-(ia+\lambda\eta_1)x-\int_0^x \mu(\tau)d\tau}dx\right.$$

$$
\begin{aligned}
&+q_2 \int_0^\infty \frac{(\lambda\eta_1 x)^3}{3!}\mu(x)e^{-(ia+\lambda\eta_1)x-\int_0^x \mu(\tau)d\tau}dx\Bigg| \\
&+\cdots\cdots \\
< &\,\eta_2 \int_0^\infty \mu(x)e^{-\lambda\eta_1 x-\int_0^x \mu(\tau)d\tau}dx \\
&+\eta_2 \int_0^\infty \lambda\eta_1 x\mu(x)e^{-\lambda\eta_1 x-\int_0^x \mu(\tau)d\tau}dx \\
&+q_2 \int_0^\infty \mu(x)e^{-\lambda\eta_1 x-\int_0^x \mu(\tau)d\tau}dx \\
&+\eta_2 \int_0^\infty \frac{(\lambda\eta_1 x)^2}{2!}\mu(x)e^{-\lambda\eta_1 x-\int_0^x \mu(\tau)d\tau}dx \\
&+q_2 \int_0^\infty \lambda\eta_1 x\mu(x)e^{-\lambda\eta_1 x-\int_0^x \mu(\tau)d\tau}dx \\
&+\eta_2 \int_0^\infty \frac{(\lambda\eta_1 x)^3}{3!}\mu(x)e^{-\lambda\eta_1 x-\int_0^x \mu(\tau)d\tau}dx \\
&+q_2 \int_0^\infty \frac{(\lambda\eta_1 x)^2}{2!}\mu(x)e^{-\lambda\eta_1 x-\int_0^x \mu(\tau)d\tau}dx \\
&+\eta_2 \int_0^\infty \frac{(\lambda\eta_1 x)^4}{4!}\mu(x)e^{-\lambda\eta_1 x-\int_0^x \mu(\tau)d\tau}dx \\
&+q_2 \int_0^\infty \frac{(\lambda\eta_1 x)^3}{3!}\mu(x)e^{-\lambda\eta_1 x-\int_0^x \mu(\tau)d\tau}dx \\
&+\cdots\cdots \\
= &\,\eta_2 \int_0^\infty \sum_{n=0}^\infty \frac{(\lambda\eta_1 x)^n}{n!}\mu(x)e^{-\lambda\eta_1 x-\int_0^x \mu(\tau)d\tau}dx \\
&+q_2 \int_0^\infty \sum_{n=0}^\infty \frac{(\lambda\eta_1 x)^n}{n!}\mu(x)e^{-\lambda\eta_1 x-\int_0^x \mu(\tau)d\tau}dx \\
= &\,\eta_2 \int_0^\infty e^{\lambda\eta_1 x}\mu(x)e^{-\lambda\eta_1 x-\int_0^x \mu(\tau)d\tau}dx \\
&+q_2 \int_0^\infty e^{\lambda\eta_1 x}\mu(x)e^{-\lambda\eta_1 x-\int_0^x \mu(\tau)d\tau}dx \\
= &\,\eta_2 \int_0^\infty \mu(x)e^{-\int_0^x \mu(\tau)d\tau}dx + q_2 \int_0^\infty \mu(x)e^{-\int_0^x \mu(\tau)d\tau}dx \\
= &\,\eta_2 + q_2 = 1.
\end{aligned}
$$

This shows the coefficient of $|a_i|$ $(i \geq 2)$ is less than 1. It is obvious that the coefficient of $|a_1|$ is less than 1. In other words, it is impossible that $\|\Phi D_\gamma\| = 1$ and therefore (4-200) holds. (4-200) and Theorem 1.27 imply the spectral radius $r(\Phi D_\gamma) \leq \|\Phi D_\gamma\| < 1$ as $|a| > \mathbb{M}$, from which together with Lemma 4.15 we know

$\gamma \in \rho(A+U+E)$ for $|a| > \mathbb{M}$. In other words, $\{ia \mid |a| > \mathbb{M}\} \subset \rho(A+U+E)$, which implies that $\sigma(A+U+E) \cap i\mathbb{R} \subset \{ia \mid |a| \leq M\}$ is bounded. On the other hand, Theorem 4.2 and Lemma 4.12 show that $s(A+U+E) = 0$, which implies $\sigma(A+U+E) \cap i\mathbb{R}$ is imaginary additively cyclic by Theorem 4.2 and Theorem 1.88, from which together with the boundedness of $\sigma(A+U+E) \cap i\mathbb{R}$ we know that $\sigma(A+U+E) \cap i\mathbb{R} = \{0\}$. $\square$

Replacing μ by $\mu(x)$ and repeating the processes of Lemma 4.8 we obtain the following result.

Lemma 4.17. *If $\eta_2 > \lambda\eta_1 \int_0^\infty e^{-\int_0^x \mu(\tau)d\tau} dx$, then 0 is an eigenvalue of $(A+U+E)^*$ with geometric multiplicity 1.*

Finally, by combining Theorem 4.2, Lemma 4.12, Lemma 4.16, Lemma 4.17 and Lemma 4.9 with Theorem 1.96 we deduce the desired result in this section.

Theorem 4.18. *Let $\mu(x) : [0,\infty) \to [0,\infty)$ be measurable and $0 < \inf\limits_{x\in[0,\infty)} \mu(x) \leq \sup\limits_{x\in[0,\infty)} \mu(x) < \infty$. If $\eta_2 > \lambda\eta_1 \int_0^\infty e^{-\int_0^x \mu(\tau)d\tau} dx$, then the time-dependent solution of the system (4-23) converges strongly to its steady-state solution, that is,*

$$\lim_{t\to\infty} p(x,t) = \langle q^*, p(0)\rangle p(x),$$

where $q^*(x)$ and $p(x)$ are eigenvectors corresponding to 0 in Lemma 4.12 and Lemma 4.17 respectively and satisfy $\langle p, q^*\rangle = 1$.

Theorem 4.11 also shows that the convergence of the time-dependent solution of the system (4-23) in Theorem 4.18 is best, that is to say, it is impossible that the time-dependent solution of the system (4-23) converges exponentially to its steady-state solution.

Remark 4.19. It is interesting that when $\eta_1 = 1$ and $\eta_2 = 1$ (equivalently $q_2 = 0$) the system of equations (4-16)–(4-21) becomes the M/G/1 queueing model which was established by Cox [18] in 1955, and when $\eta_1 = 1$, $\eta_2 = 1$ (equivalently $q_2 = 0$) and $\mu(x) = \mu$ the system of equations (4-16)–(4-21) becomes the M/M/1 queueing model which was introduced and discussed in detail in Gupur et al. [65]

4.5 Asymptotic Behavior of Some Reliability Indices of the System (4-23)

According to Li and Cao [78] the system has four reliability indices. $W_{r,1}(t)$, the reprocessed rate of machine 1 at time t, is defined by

$$W_{r,1}(t) = \lambda q_1 p_0(t) + \lambda q_1 \sum_{n=1}^{\infty} \int_0^\infty p_n(x,t)dx. \tag{4-201}$$

$W_{0,1}(t)$, the output rate of machine 1 at time t, is denoted by

$$W_{0,1}(t) = \lambda\eta_1 p_0(t) + \lambda\eta_1 \sum_{n=1}^{\infty} \int_0^{\infty} p_n(x,t)dx. \tag{4-202}$$

$W(t)$, the production rate of the system at time t, is defined by

$$W(t) = \eta_2 \sum_{n=1}^{\infty} \int_0^{\infty} p_n(x,t)\mu(x)dx. \tag{4-203}$$

$W_{r,2}(t)$, the reprocessed rate of machine 2 at time t, is denoted by

$$W_{r,2}(t) = q_2 \sum_{n=1}^{\infty} \int_0^{\infty} p_n(x,t)\mu(x)dx. \tag{4-204}$$

In the following we will discuss asymptotic behavior of the above indices.

Theorem 4.20. *Let $\mu(x) : [0,\infty) \to [0,\infty)$ be measurable and $0 < \inf\limits_{x\in[0,\infty)} \mu(x) \leq \sup\limits_{x\in[0,\infty)} \mu(x) < \infty$. If $\eta_2 > \lambda\eta_1 \int_0^{\infty} e^{-\int_0^x \mu(\tau)d\tau}dx$, then*

$$\lim_{t\to\infty} W_{r,1}(t) = \frac{\lambda q_1 \eta_2}{\eta_2 - \lambda\eta_1 \int_0^{\infty} e^{-\int_0^x \mu(\tau)d\tau}dx} := W_{r,1}, \tag{4-205}$$

$$\lim_{t\to\infty} W_{0,1}(t) = \frac{\lambda \eta_1 \eta_2}{\eta_2 - \lambda\eta_1 \int_0^{\infty} e^{-\int_0^x \mu(\tau)d\tau}dx} := W_{0,1}, \tag{4-206}$$

$$\lim_{t\to\infty} W(t) = \frac{\lambda \eta_1 \eta_2}{\eta_2 - \lambda\eta_1 \int_0^{\infty} e^{-\int_0^x \mu(\tau)d\tau}dx} := W, \tag{4-207}$$

$$\lim_{t\to\infty} W_{r,2}(t) = \frac{\lambda \eta_1 q_2}{\eta_2 - \lambda\eta_1 \int_0^{\infty} e^{-\int_0^x \mu(\tau)d\tau}dx} := W_{r,2}. \tag{4-208}$$

Proof. If we define

$$Y = \left\{ p \in X \;\middle|\; \begin{array}{l} p(x) = (p_0, p_1(x), p_2(x), \dots), \\ p_0 \geq 0,\ p_n(x) \geq 0,\ x \in [0,\infty) \end{array} \right\},$$

then by Kelley et al. [72] Y is a cone. In Y we introduce an order relation as

$$p \leq y \Longleftrightarrow p_0 \leq y_0,\ p_n(x) \leq y_n(x),\ n \geq 1,\ x \in [0,\infty),\ p,\ y \in Y.$$

Under this relation Y becomes a partial order set, see Kelley et al. [72]. It is obvious that the initial value $p(0) = (1, 0, 0, \dots)$ of the system (4-23) and the eigenvector $p(x)$ corresponding to 0 belong to Y when $p_0 \geq 0$ (see Lemma 4.12). If we take $p_0 = 1$ in Lemma 4.12, then by (4-166), (4-167), (4-168), (4-169) and (4-170) we have

$$p(0) \leq p(x). \tag{4-209}$$

Since $T(t)$ is a linear positive operator by Theorem 4.2, $T(t)$ is a monotone increasing operator on Y. Hence, by (4-209) and Theorem 4.3 it follows that

$$p(x,t) = T(t)p(0) \le T(t)p(x) = p(x). \tag{4-210}$$

Here we have used the relation $T(t)p(x) = p(x)$, which follows from Lemma 4.12. (4-210) implies

$$\begin{aligned}
&p_0(t) \le 1,\ p_n(x,t) \le p_n(x),\ (x,t) \in [0,\infty)\times[0,\infty)\\
&\Longrightarrow\\
&p_0(t) \le 1,\ \int_0^\infty p_n(x,t)dx \le \int_0^\infty p_n(x)dx,\ n \ge 1,\ t \ge 0.
\end{aligned} \tag{4-211}$$

Since $p_n(x,t)$ and $p_n(x)$ are positive, it follows that

$$W_{r,1}(t) = \lambda q_1 p_0(t) + \lambda q_1 \sum_{n=1}^\infty \int_0^\infty p_n(x,t)dx \le \lambda q_1 + \lambda q_1 \sum_{n=1}^\infty \int_0^\infty p_n(x)dx. \tag{4-212}$$

And (4-173) gives

$$\lambda q_1 + \lambda q_1 \sum_{n=1}^\infty \int_0^\infty p_n(x)dx = \lambda q_1 + \frac{\lambda\eta_1 \int_0^\infty e^{-\int_0^x \mu(\tau)d\tau}dx}{\eta_2 - \lambda\eta_1\int_0^\infty e^{-\int_0^x \mu(\tau)d\tau}dx} < \infty. \tag{4-213}$$

Theorem 4.18 implies

$$\begin{aligned}
&\lim_{t\to\infty}\left\{|p_0(t)-p_0| + \sum_{n=1}^\infty \int_0^\infty |p_n(x,t)-p_n(x)|dx\right\} = 0\\
&\Longrightarrow\\
&\lim_{t\to\infty}|p_0(t)-p_0| = 0,\\
&\lim_{t\to\infty}\int_0^\infty |p_n(x,t)-p_n(x)|dx = 0, \quad n \ge 1.
\end{aligned} \tag{4-214}$$

From this together with the dominated convergence theorem (see Theorem 1.47), (4-212), (4-213) and (4-214) we have

$$\begin{aligned}
W_{r,1} &:= \lim_{t\to\infty} W_{r,1}(t) = \lim_{t\to\infty}\lambda q_1 p_0(t) + \lambda q_1 \lim_{t\to\infty}\sum_{n=1}^\infty \int_0^\infty p_n(x,t)dx\\
&= \lambda q_1 p_0 + \lambda q_1 \sum_{n=1}^\infty \lim_{t\to\infty}\int_0^\infty p_n(x,t)dx = \lambda q_1 + \lambda q_1\sum_{n=1}^\infty \int_0^\infty p_n(x)dx\\
&= \lambda q_1 + \lambda q_1 \frac{\lambda\eta_1\int_0^\infty e^{-\int_0^x \mu(\tau)d\tau}dx}{\eta_2 - \lambda\eta_1\int_0^\infty e^{-\int_0^x \mu(\tau)d\tau}dx} = \frac{\lambda q_1\eta_2}{\eta_2 - \lambda\eta_1\int_0^\infty e^{-\int_0^x \mu(\tau)d\tau}dx}.
\end{aligned}$$

Similarly,

$$\begin{aligned}
W_{0,1} &:= \lim_{t\to\infty} W_{0,1}(t) \\
&= \lim_{t\to\infty} \lambda\eta_1 p_0(t) + \lim_{t\to\infty} \lambda\eta_1 \sum_{n=1}^{\infty} \int_0^{\infty} p_n(x,t)dx \\
&= \frac{\lambda\eta_1\eta_2}{\eta_2 - \lambda\eta_1 \int_0^{\infty} e^{-\int_0^x \mu(\tau)d\tau} dx}.
\end{aligned}$$

(4-214) implies

$$\begin{aligned}
0 &\le \lim_{t\to\infty} \int_0^{\infty} |\mu(x)p_n(x,t) - \mu(x)p_n(x)|dx \\
&\le \lim_{t\to\infty} \sup_{x\in[0,\infty)} \mu(x) \int_0^{\infty} |p_n(x,t) - p_n(x)|dx \\
&= \sup_{x\in[0,\infty)} \mu(x) \lim_{t\to\infty} \int_0^{\infty} |p_n(x,t) - p_n(x)|dx = 0 \\
&\Longrightarrow \\
&\lim_{t\to\infty} \int_0^{\infty} |\mu(x)p_n(x,t) - \mu(x)p_n(x)|dx = 0, \quad n \ge 1.
\end{aligned}$$

From this together with (4-173), (4-211) and the dominated convergence theorem (see Theorem 1.47) we deduce the asymptotic behavior of the other two indices.

$$\begin{aligned}
W &:= \lim_{t\to\infty} W(t) = \lim_{t\to\infty} \eta_2 \sum_{n=1}^{\infty} \int_0^{\infty} p_n(x,t)\mu(x)dx \\
&= \frac{\lambda\eta_1\eta_2 \int_0^{\infty} \mu(x) e^{-\int_0^x \mu(\tau)d\tau} dx}{\eta_2 - \lambda\eta_1 \int_0^{\infty} e^{-\int_0^x \mu(\tau)d\tau} dx} \\
&= \frac{\lambda\eta_1\eta_2}{\eta_2 - \lambda\eta_1 \int_0^{\infty} e^{-\int_0^x \mu(\tau)d\tau} dx}.
\end{aligned}$$

$$\begin{aligned}
W_{r,2} &:= \lim_{t\to\infty} W_{r,2}(t) = q_2 \sum_{n=1}^{\infty} \int_0^{\infty} p_n(x,t)\mu(x)dx \\
&= \frac{\lambda\eta_1 q_2}{\eta_2 - \lambda\eta_1 \int_0^{\infty} e^{-\int_0^x \mu(\tau)d\tau} dx}.
\end{aligned}$$

□

Chapter 5

Further Research Problems

In 1994, Li and Cao [79] established a mathematical model describing stochastic scheduling on an unreliable machine with general uptime and general set-up times as follows.

$$\begin{aligned}
&\frac{\partial p_0(x,t)}{\partial t} + \frac{\partial p_0(x,t)}{\partial x} = -(\lambda(x) + \mu(x))p_0(x,t),\\
&\frac{\partial p_1(x,t)}{\partial t} + \frac{\partial p_1(x,t)}{\partial x} = -\eta(x)p_1(x,t),\\
&p_0(0,t) = \delta(t) + \int_0^\infty p_1(x,t)\eta(x)dx,\\
&p_1(0,t) = \int_0^\infty p_0(x,t)\lambda(x)dx\\
&p_0(x,0) = \delta(x), \quad p_1(x,0) = 0,
\end{aligned}$$

where $(x,t) \in [0,\infty) \times [0,\infty)$, $p_0(x,t)dx$ represents the probability that at time t the machine is processing job i and the elapsed processing time of job i lies in $[x, x+dx)$, $p_1(x,t)dx$ represents the probability that at time t the machine has been subject to breakdown and is being repaired and the elapsed repair time lies in $[x, x+dx)$, $\delta(\cdot)$ is the Dirac delta function, $\mu(x), \lambda(x), \eta(x)$ are the hazard rate functions of random processing time of job i, the random uptime of job i, and random repair time respectively.

Well-posedness of the above system of equations and asymptotic property of its time-dependent solution have not been obtained until now (see Xu [109], pages 565–567).

In Su [102], Baohe Su established the following system of equations by using the supplementary variable technique:

$$\begin{aligned}
&\frac{\partial p_{0,0,k}(x,t)}{\partial t} + \frac{\partial p_{0,0,k}(x,t)}{\partial x}\\
&= -(\lambda_{0,1} + \lambda_{1,0} + \alpha(x))p_{0,0,k}(x,t),\ k \geq 0,
\end{aligned}$$

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$$\frac{\partial p_{0,1,0}(x,t)}{\partial t} + \frac{\partial p_{0,1,0}(x,t)}{\partial x}$$
$$= -(\lambda_{1,0} + \lambda_{0,2} + \alpha(x))p_{0,1,0}(x,t) + \sum_{k=0}^{\infty} \lambda_{0,1} p_{0,0,k}(x,t),$$
$$\frac{\partial p_{1,0,0}(x,t)}{\partial t} + \frac{\partial p_{1,0,0}(x,t)}{\partial x}$$
$$= -(\lambda_{0,1} + \lambda_{2,0} + \alpha(x))p_{1,0,0}(x,t) + \sum_{k=0}^{\infty} \lambda_{1,0} p_{0,0,k}(x,t),$$
$$\frac{\partial p_{0,1,1}(x,t)}{\partial t} + \frac{\partial p_{0,1,1}(x,t)}{\partial x} = -\mu_{0,1}(x)p_{0,1,1}(x,t),$$
$$\frac{\partial p_{1,0,1}(x,t)}{\partial t} + \frac{\partial p_{1,0,1}(x,t)}{\partial x} = -\mu_{1,0}(x)p_{1,0,1}(x,t),$$
$$\frac{\partial p_{0,2,0}(x,t)}{\partial t} + \frac{\partial p_{0,2,0}(x,t)}{\partial x} = -\mu_{0,2}(x)p_{0,2,0}(x,t),$$
$$\frac{\partial p_{2,0,0}(x,t)}{\partial t} + \frac{\partial p_{2,0,0}(x,t)}{\partial x} = -\mu_{2,0}(x)p_{2,0,0}(x,t),$$
$$\frac{\partial p_{1,1,0}(x,t)}{\partial t} + \frac{\partial p_{1,1,0}(x,t)}{\partial x}$$
$$= -(\lambda_{0,2} + \lambda_{2,0} + \alpha(x))p_{1,1,0}(x,t) + \lambda_{1,0} p_{0,1,0}(x,t)$$
$$+ \lambda_{0,1} p_{1,0,0}(x,t),$$
$$\frac{\partial p_{1,1,1}(x,t)}{\partial t} + \frac{\partial p_{1,1,1}(x,t)}{\partial x} = -\mu_{1,1}(x)p_{1,1,1}(x,t),$$
$$\frac{\partial p_{1,2,0}(x,t)}{\partial t} + \frac{\partial p_{1,2,0}(x,t)}{\partial x} = -\mu_{1,2}(x)p_{1,2,0}(x,t),$$
$$\frac{\partial p_{2,1,0}(x,t)}{\partial t} + \frac{\partial p_{2,1,0}(x,t)}{\partial x} = -\mu_{2,1}(x)p_{2,1,0}(x,t),$$
$$p_{0,0,0}(0,t) = \delta(t) + \int_0^{\infty} \mu_{0,1}(x)p_{0,1,1}(x,t)dx$$
$$+ \int_0^{\infty} \mu_{1,0}(x)p_{1,0,1}(x,t)dx$$
$$+ \int_0^{\infty} \mu_{0,2}(x)p_{0,2,0}(x,t)dx$$
$$+ \int_0^{\infty} \mu_{2,0}(x)p_{2,0,0}(x,t)dx$$
$$+ \int_0^{\infty} \mu_{1,2}(x)p_{1,2,0}(x,t)dx$$
$$+ \int_0^{\infty} \mu_{2,1}(x)p_{2,1,0}(x,t)dx$$

$$
\begin{aligned}
&+\int_0^\infty \mu_{1,1}(x)p_{1,1,1}(x,t)dx,\\
&p_{0,0,k}(0,t)=\int_0^\infty \alpha(x)p_{0,0,k-1}(x,t)dx,\ k\geq 1,\\
&p_{0,1,0}(0,t)=0,\\
&p_{1,0,0}(0,t)=0,\\
&p_{0,1,1}(0,t)=\int_0^\infty \alpha(x)p_{0,1,0}(x,t)dx,\\
&p_{1,0,1}(0,t)=\int_0^\infty \alpha(x)p_{1,0,0}(x,t)dx,\\
&p_{0,2,0}(0,t)=\int_0^\infty \lambda_{0,2}p_{0,1,0}(x,t)dx,\\
&p_{2,0,0}(0,t)=\int_0^\infty \lambda_{2,0}p_{1,0,0}(x,t)dx,\\
&p_{1,1,0}(0,t)=0,\\
&p_{1,1,1}(0,t)=\int_0^\infty \alpha(x)p_{1,1,0}(x,t)dx,\\
&p_{1,2,0}(0,t)=\int_0^\infty \lambda_{0,2}p_{1,1,0}(x,t)dx,\\
&p_{2,1,0}(0,t)=\int_0^\infty \lambda_{2,0}p_{1,1,0}(x,t)dx,\\
&p_{0,0,0}(x,0)=\delta(x),\\
&p_{0,0,k}(x,0)=0,\quad k\geq 1,\\
&p_{1,0,0}(x,0)=0,\\
&p_{0,1,0}(x,0)=0,\\
&p_{0,1,1}(x,0)=0,\\
&p_{1,0,1}(x,0)=0,\\
&p_{0,2,0}(x,0)=0,\\
&p_{2,0,0}(x,0)=0,\\
&p_{1,1,0}(x,0)=0,\\
&p_{1,1,1}(x,0)=0,\\
&p_{1,2,0}(x,0)=0,\\
&p_{2,1,0}(x,0)=0.
\end{aligned}
$$

Here, $\lambda_{1,0}$ is the failure rate of unit 1 from normal to partial failure; $\lambda_{2,0}$ is the failure rate of unit 1 from partial failure to total failure; $\lambda_{0,1}$ is the failure rate of unit 2 from normal to partial failure; $\lambda_{0,2}$ is the failure rate of unit 2 from partial failure to total failure; $\alpha(x)$ is the check rate satisfying $\alpha(x)\geq$

0, $\int_0^\infty \alpha(x)dx = \infty$; $\mu_{i,0}(x)$ $(i = 1, 2)$ is the repair rate of unit 1 satisfying $\mu_{i,0}(x) \geq 0$, $\int_0^\infty \mu_{i,0}(x)dx = \infty$; $\mu_{0,i}(x)$ $(i = 1, 2)$ is the repair time of unit 2 satisfying $\mu_{0,i}(x) \geq 0$, $\int_0^\infty \mu_{0,i}(x)dx = \infty$; $\mu_{i,j}(x)$ $(i, j = 1, 2)$ is the repair rate of both the units satisfying $\mu_{i,j}(x) \geq 0$, $\int_0^\infty \mu_{i,j}(x)dx = \infty$; $p_{0,0,k}(x, t)$ $(k \geq 0)$ is the probability that unit 1 is normal, unit 2 is normal, the system has been checked k times and has an elapsed time of x; $p_{1,0,0}(x, t)$ is the probability that unit 1 is partially failed, unit 2 is normal, the system has not been checked and has an elapsed time of x; $p_{0,1,0}(x, t)$ is the probability that unit 1 is normal, unit 2 is partially failed, the system has not been checked and has an elapsed time of x; $p_{1,1,0}(x, t)$ is the probability that unit 1 is partially failed, unit 2 is partially failed, the system has not been checked and has an elapsed time of x; $p_{0,1,1}(x, t)$ is the probability that unit 1 is normal, unit 2 is partially failed, the system has been checked one time and has an elapsed repair time of x; $p_{1,0,1}(x, t)$ is the probability that unit 1 is partially failed, unit 2 is normal, the system has been checked one time and has an elapsed repair time of x; $p_{0,2,0}(x, t)$ is the probability that unit 1 is normal, unit 2 is total failure and the system has not been checked; $p_{2,0,0}(x, t)$ is the probability that unit 1 is a total failure, unit 2 is normal and the system has not been checked; $p_{1,1,1}(x, t)$ is the probability that unit 1 is partially failed, unit 2 is partially failed, the system has been checked one time and has an elapsed repair time of x; $p_{1,2,0}(x, t)$ is the probability that unit 1 is partially failed, unit 2 is totally failed, the system has not been checked and has an elapsed repair time of x; $p_{2,1,0}(x, t)$ is the probability that unit 1 is totally failed, unit 2 is partially failed, the system has not been checked and has an elapsed repair time of x.

Well-posedness of the system of the above equations and asymptotic behavior of its time-dependent solution have not been obtained until now.

In Li et al. [80], Li and Shi established the following reliability model:

$$
\begin{aligned}
\frac{\partial p_{0,1}(x,t)}{\partial t} + \frac{\partial p_{0,1}(x,t)}{\partial x} &= -(a_1 + a_2(x))p_{0,1}(x,t) \\
&\quad + \int_0^x p_2(x,u,t)b_1(u)du, \\
\frac{\partial p_{0,i}(x,t)}{\partial t} + \frac{\partial p_{0,i}(x,t)}{\partial x} &= -(a_1 + a_2(x))p_{0,i}(x,t) + a_1 p_{0,i-1}(x,t), \\
&\quad 2 \leq i \leq m, \\
\frac{\partial p_{1,1}(x,t)}{\partial t} + \frac{\partial p_{1,1}(x,t)}{\partial x} &= -(a_1 + b_2(x))p_{1,1}(x,t) \\
&\quad + \int_0^\infty p_4(x,u,t)b_1(u)du, \\
\frac{\partial p_{1,i}(x,t)}{\partial t} + \frac{\partial p_{1,i}(x,t)}{\partial x} &= -(a_1 + b_2(x))p_{1,i}(x,t) \\
&\quad + a_1 p_{1,i-1}(x,t),\ 2 \leq i \leq m,
\end{aligned}
$$

$$\frac{\partial p_2(x,u,t)}{\partial t} + \frac{\partial p_2(x,u,t)}{\partial x} + \frac{\partial p_2(x,u,t)}{\partial u}$$
$$= -(a_2(x) + b_1(u))p_2(x,u,t),$$

$$\frac{\partial p_3(x,t)}{\partial t} + \frac{\partial p_3(x,t)}{\partial x} = -b_1(x)p_3(x,t)$$
$$+ \int_x^\infty a_2(u)p_2(x,u,t)du,$$

$$\frac{\partial p_4(x,u,t)}{\partial t} + \frac{\partial p_4(x,u,t)}{\partial x} = -b_1(x)p_4(x,u,t),$$

$$p_{0,1}(0,t) = \int_0^\infty p_{1,1}(x,t)b_2(x)dx + \delta(t),$$

$$p_{0,i}(0,t) = \int_0^\infty p_{1,i}(x,t)b_2(x)dx, \quad 2 \le i \le m,$$

$$p_{1,1}(0,t) = \int_0^\infty p_3(x,t)b_1(x)dx + \int_0^\infty p_{0,1}(x,t)a_2(x)dx,$$

$$p_{1,i}(0,t) = \int_0^\infty p_{0,i}(x,t)a_2(x)dx,\ 2 \le i \le m,$$

$$p_2(0,u,t) = a_1 p_{0,m}(u,t),$$

$$p_3(0,t) = 0,$$

$$p_4(0,u,t) = a_1 p_{1,m}(u,t),$$

$$p_{0,1}(x,0) = \delta(x),\ p_{0,i}(x,0) = 0,\ 2 \le i \le m,$$

$$p_{1,i}(x,0) = 0, \quad 0 \le i \le m,$$

$$p_2(x,u,0) = p_3(x,u,0) = p_4(x,u,0) = 0.$$

Where $(x,u,t) \in [0,\infty) \times [0,\infty) \times [0,\infty)$; $p_{0,i}(x,t)$ is the probability that at time t two units are operating and the operating unit 1 is in phase i, and the age of unit 2 is $x,\ i = 1,2,\ldots,m$; $p_{1,i}(x,t)$ is the probability that at time t unit 1 is operating, the operating phase is i and unit 2 is being repaired and the elapsed time of the failed unit 2 is $x,\ i = 1,2,\ldots,m$; $p_2(x,u,t)$ is the probability that at time t unit 2 is operating and unit 1 is being repaired, the elapsed repair time of the failed unit 1 is x, and the age of unit 2 is $u,$; $p_3(x,t)$ is the probability that at time t unit 1 is being repaired and unit 2 is waiting for repair, and the elapsed repair time of the failed unit 1 is x; $p_4(x,u,t)$ is the probability that at time t unit 1 is being repaired and unit 2 is in "suspended repair", and the elapsed repair time of the failed unit 1 is x and the elapsed time of the failed unit 2 is u; $a_1 > 0$ is a constant parameter of the Erlang distribution; $a_2(x)$ is the repair rate of unit 1; $b_1(x)$ is the life rate; $b_2(x)$ is the repair rate of unit 2; $\delta(\cdot)$ is the Dirac-delta function.

Well-posedness of the above model and asymptotic behavior of its time-dependent solution have not been obtained until today.

In 1976, by using the supplementary variable technique, Linton [81] described two-unit parallel redundant systems through the following equations:

$$\begin{aligned}
&\frac{\partial q_{0,1}(x,t)}{\partial t}+\frac{\partial q_{0,1}(x,t)}{\partial x}=\mu q_{2,m}(x,t),\\
&\frac{\partial q_{0,i}(x,t)}{\partial t}+\frac{\partial q_{0,i}(x,t)}{\partial x}=-\lambda q_{0,i}(x,t)\\
&\qquad\qquad\qquad\qquad\qquad\quad+\lambda q_{0,i-1}(x,t),\quad 2\le i\le k,\\
&\frac{\partial q_{1,1}(x,t)}{\partial t}+\frac{\partial q_{1,1}(x,t)}{\partial x}=0,\\
&\frac{\partial q_{1,2}(x,t)}{\partial t}+\frac{\partial q_{1,2}(x,t)}{\partial x}=-\lambda q_{1,2}(x,t)+\lambda e^{-\lambda y}q_{1,1}(x,t),\\
&\frac{\partial q_{1,i}(x,t)}{\partial t}+\frac{\partial q_{1,i}(x,t)}{\partial x}=-\lambda q_{1,i}(x,t)\\
&\qquad\qquad\qquad\qquad\qquad\quad+\lambda q_{1,i-1}(x,t),\quad 3\le i\le k,\\
&\frac{\partial q_{2,1}(x,t)}{\partial t}+\frac{\partial q_{2,1}(x,t)}{\partial x}=-\mu q_{2,1}(x,t)+\lambda q_{0,k}(x,t),\\
&\frac{\partial q_{2,j}(x,t)}{\partial t}+\frac{\partial q_{2,j}(x,t)}{\partial x}=-\mu q_{2,j}(x,t)\\
&\qquad\qquad\qquad\qquad\qquad\quad+\mu q_{2,j-1}(x,t),\quad 2\le j\le m,\\
&q_{0,1}(0,t)=\int_0^\infty q_{1,1}(x,t)g(x)dx+\delta(t),\\
&q_{0,i}(0,t)=\int_0^\infty q_{1,i}(x,t)g(x)dx,\quad 2\le i\le k,\\
&q_{1,i}(0,t)=\int_0^\infty q_{0,i}(x,t)f(x)dx,\quad 1\le i\le k,\\
&q_{2,j}(0,t)=0,\quad 1\le j\le m,\\
&q_{0,1}(x,0)=\delta(x),\quad q_{0,i}(x,0)=0,\quad 2\le i\le k\\
&q_{1,i}(x,0)=0,\ 1\le i\le k;\ q_{2,j}(x,0)=0,\ 1\le j\le m.
\end{aligned}$$

Where $(x,t)\in[0,\infty)\times[0,\infty)$; $p_{0,i}(x,t)$ $(1\le i\le k)$ is the probability that at moment t, no units are down, unit 2 is in the ith stage of failure, and unit 1 has been operating for an amount of time x; $p_{1,i}(x,t)$ $(1\le i\le k)$ is the probability that at moment t, unit 1 is down, unit 2 is in the ith stage of failure and an amount of x has already been spent on the current repair of unit 1; $p_{2,j}(x,t)$ $(1\le j\le m)$ are the probability that at moment t, unit 2 is down, unit 2 is in the jth phase of repair, and unit 1 has been operating for an amount of time x; $\lambda>0$ and $\mu>0$ are parameters of unit 1 and unit 2 respectively; $g(x)$ is the repair rate of unit 1; $f(x)$ is the failure rate of unit 2, $\delta(\cdot)$ is the Dirac-delta function.

Well-posedness of the above model and asymptotic behavior of its time-dependent solution have not been obtained until today.

In 2006, Yue et al. [116] studied a Gaver's parallel system and established the following two models:

$$
\begin{aligned}
&\frac{\partial Q_0(x,t)}{\partial t}+\frac{\partial Q_0(x,t)}{\partial x}=-(2\lambda+\alpha(x))Q_0(x,t),\\
&\frac{\partial Q_1(x,t)}{\partial t}+\frac{\partial Q_1(x,t)}{\partial x}=-(2\lambda+\alpha(x))Q_1(x,t)+2\lambda Q_0(x,t),\\
&\frac{\partial Q_2(x,t)}{\partial t}+\frac{\partial Q_2(x,t)}{\partial x}=-(2\lambda+\alpha(x))Q_2(x,t)+2\lambda Q_1(x,t),\\
&\frac{\partial Q_3(x,t)}{\partial t}+\frac{\partial Q_3(x,t)}{\partial x}=-(2\lambda+\mu(x))Q_3(x,t),\\
&\frac{\partial Q_4(x,t)}{\partial t}+\frac{\partial Q_4(x,t)}{\partial x}=-(\lambda+\mu(x))Q_4(x,t)+2\lambda Q_3(x,t),\\
&Q_0(0,t)=\int_0^\infty \alpha(x)Q_0(x,t)dx+\int_0^\infty \mu(x)Q_3(x,t)dx+\delta(t),\\
&Q_1(0,t)=Q_2(0,t)=0,\\
&Q_3(0,t)=\int_0^\infty \alpha(x)Q_1(x,t)dx+\int_0^\infty \mu(x)Q_4(x,t)dx,\\
&Q_4(0,t)=\int_0^\infty \alpha(x)Q_2(x,t)dx,\\
&Q_0(x,0)=\delta(x),\ Q_i(x,0)=0,\ i=1,2,3,4.
\end{aligned}
$$

Here $(x,t)\in[x,\infty)\times[0,\infty)$; $Q_i(x,t)$ $(i=0,1,2)$ is the probability that at time t the system is normal and the elapsed vacation time of the repairman is x; $Q_i(x,t)$ $(i=3,4)$ is the probability that at time t the system is down and the elapsed repair time of the piece is x; λ is the working rate, $\mu(x)$ is the repair rate, $\alpha(x)$ is the vacation rate of the repairman, $\delta(\cdot)$ is the Dirac-delta function.

$$
\begin{aligned}
&\frac{\partial P_0(x,t)}{\partial t}+\frac{\partial P_0(x,t)}{\partial x}=-(2\lambda+\alpha(x))P_0(x,t),\\
&\frac{\partial P_1(x,t)}{\partial t}+\frac{\partial P_1(x,t)}{\partial x}=-(2\lambda+\alpha(x))P_1(x,t)+2\lambda P_0(x,t),\\
&\frac{\partial P_2(x,t)}{\partial t}+\frac{\partial P_2(x,t)}{\partial x}=-(\lambda+\alpha(x))P_2(x,t)+2\lambda P_1(x,t),\\
&\frac{\partial P_3(x,t)}{\partial t}+\frac{\partial P_3(x,t)}{\partial x}=-(2\lambda+\mu(x))P_3(x,t),\\
&\frac{\partial P_4(x,t)}{\partial t}+\frac{\partial P_4(x,t)}{\partial x}=-(\lambda+\mu(x))P_4(x,t)+2\lambda P_3(x,t),\\
&\frac{\partial P_5(x,t)}{\partial t}+\frac{\partial P_5(x,t)}{\partial x}=-\alpha(x))P_5(x,t)+\lambda P_2(x,t),\\
&\frac{\partial P_6(x,t)}{\partial t}+\frac{\partial P_6(x,t)}{\partial x}=-\mu(x)P_6(x,t)+\lambda P_4(x,t),
\end{aligned}
$$

$$
\begin{aligned}
&P_0(0,t) = \int_0^\infty \alpha(x)P_0(x,t)dx + \int_0^\infty \mu(x)P_3(x,t)dx + \delta(t),\\
&P_1(0,t) = P_2(0,t) = P_5(0,t) = 0,\\
&P_3(0,t) = \int_0^\infty \alpha(x)P_1(x,t)dx + \int_0^\infty \mu(x)P_4(x,t)dx,\\
&P_4(0,t) = \int_0^\infty \alpha(x)P_2(x,t)dx + \int_0^\infty \mu(x)P_6(x,t)dx,\\
&P_6(0,t) = \int_0^\infty \alpha(x)P_5(x,t)dx,\\
&P_0(x,0) = \delta(x),\ P_i(x,0) = 0, \quad i = 1,2,3,4,5,6.
\end{aligned}
$$

Here $(x,t) \in [x,\infty) \times [0,\infty)$; $P_i(x,t)$ $(i = 0,1,2,5)$ is the probability that at time t the system is normal and the elapsed vacation time of the repairman is x; $P_i(x,t)$ $(i = 3,4,6)$ is the probability that at time t the system is down and the elapsed repair time of the piece is x; λ is the working rate, $\mu(x)$ is the repair rate, $\alpha(x)$ is the vacation rate of the repairman, $\delta(\cdot)$ is the Dirac-delta function.

Well-posedness of the system of the above equations and asymptotic behavior of their time-dependent solution have not been obtained until now.

Readers may note the main difficult point of all the above models are the Dirac-delta functions. First we need to know the meaning of their time-dependent solution, that is, whether there exists a weak/ mild/ strong time-dependent solution, then should understand convergence of their time-dependent solution as time tends to infinity, i.e., we should know weak/ weak*/ strong/ exponential convergence.

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Index